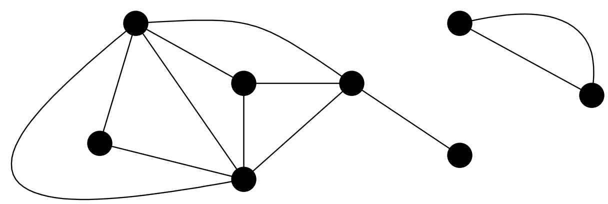
Math 145 **Fall 2003**

Graphs

Theory and examples

Basic terminology:

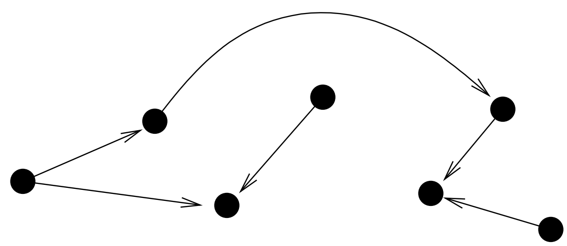
A **graph** is an object consisting of a set of points called **vertices**, some of which are connected by lines (or arcs) called **edges**.



A graph is **simple** if any 2 vertices are connected by at most one edge and there are no loops (edges starting and ending at the same vertex).

Usually the edges of a graph are not oriented.

But if the edges are oriented, then we have an **oriented** or **directed** graph. An example of an oriented graph is a one-way road system.

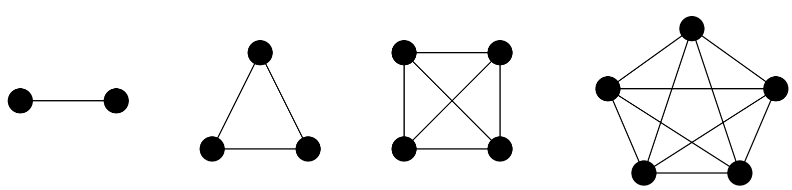


If an edge e connects the vertices v_1 and v_2, then we say that v_1 and v_2 are the **endpoints** of e. Also, we say that v_1 and v_2 are **adjacent vertices**. If two edges e_1 and e_2 share a common vertex, then we say that e_1 and e_2 are **adjacent edges**. A vertex v has **degree** m if m edges end at v.

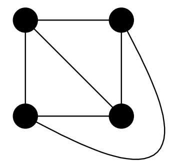
Theorem. In any graph, the sum of the degrees of the vertices equals twice the number of the edges.

Corollary. In any graph, the number of vertices with odd degrees is even.

In an undirected graph, if there are edges between every two vertices in that graph, then it is called a **complete** graph. K_n denotes the complete graph with n vertices. The graphs K_2, K_3, K_4, and K_5 are shown below.

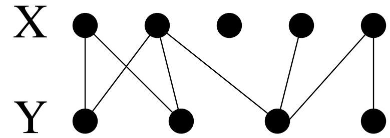


We say that a graph can be **embedded** into a plane if it is possible to draw it in such a way that no two edges intersect. For example, the graph K_4 can be embedded as follows:

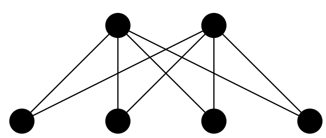


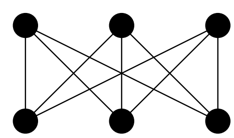
It can be proved that K_5 and $K_{3,3}$ (and many other graphs) can not be embedded into a plane.

If the vertices of a graph can be separated into two parts X and Y so that for every edge in the graph, one of its endpoints belongs to X abd the other belongs to Y, then we call this kind of graph a **bipartite** graph.



If every vertex in the set X is connected to every vertex in the set Y, then the graph is called a completete bipartite graph. $K_{m,n}$ denotes the complete bipartite graph with m vertices in the set X and n vertices in the set Y. The graphs $K_{2,4}$ and $K_{3,3}$ are shown below.





A **path** is a sequence of edges $e_1, e_2, \ldots, e_n$ such that $e_1 = (x_0, x_1)$, $e_2 = (x_1, x_2)$, $\ldots$, $e_n = (x_{n-1}, x_n)$. When there are no multiple edges in the graph, this path is denoted by its vertex sequence $x_0,\ x_1,\ \ldots, x_n$. A path that begins and ends at the same vertex is called a **cycle**. A path is **simple** if it does not contain the same edge more than once.

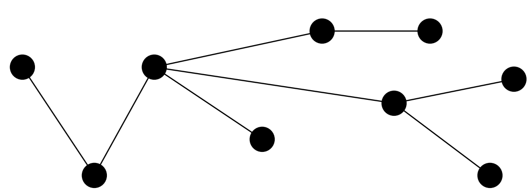
An **Euler path (Euler cycle)** is a simple path (cycle) containing every edge of the graph.

Theorem. A connected graph has an Euler cycle if and only if each of its vertices has even degree.

A **Hamilton path (Hamilton cycle)** is a simple path (cycle) visiting every vertex exactly once.

If you can visit all vertices by walking on edges, the graph is **connected**.

A connected graph without cycles is called a **tree**. Here is an example of a tree:

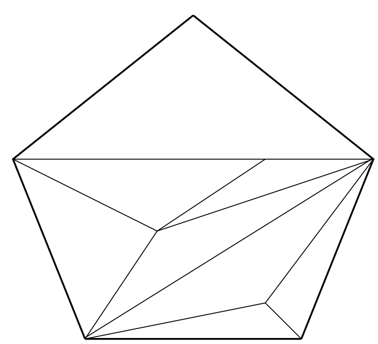


Problem. Prove that in any collection of six people either three of them mutually know each other or three of them mutually do not know each other.

Solution. Let's translate this problem into a graph theory problem. Let six vertices a, b, c, d, e, and f represent the six people. If two people know each other, then we use a red edge to join these two vertices. If two people do not know eath other, then we use a blue edge to join these two vertices. Since there are edges between every two vertices in the graph, it's a complete graph K_6 with red or blue edges. Now the problem has been translated into the following problem: We use red or blue colors to color the edges in the complete graph K_6. Prove that there must exist either 3 vertices such that the edges joining them are all red, or 3 vertices such that the edges joining them are all blue. Now, let's pick any vertex in K_6, say a. The 5 edges between this vertex and the other 5 vertices are either red or blue. According to Dirichlet's Principle, at least 3 edges of the five have the same color. Let's assume that ab, ac, ad are red (the blue case is similar). Now consider the triangle bcd. If one of the edges bc, bd, cd is red, then we have a red triangle. Otherwise, if bc, bd, cd are all blue, then the triangle bcd is a blue triangle. This proves that there must exist a triangle all of whose edges are colored by the same color.

Problem. Is it possible to draw a triangular map inside a pentagon so that the degree of each vertex is even?

Below is an example of a triangular map (but some vertices have an odd degree):



Solution. The answer is no. We will prove this by contradiction. Suppose such a map exists. We know (see a homework problem from the induction section) that every map with all vertices of even degree admits a proper coloring, i.e. its regions can be colored with 2 colors so that no neighbouring regions have the same color. Color our map with blue and red so that the (infinite) region outside of the pentagon is blue. All the other regions are triangles. Each edge has a red triangle on one side and and a blue region (either a triangle or that infinite outside region) on the other side. Now, count the number of edges (boundaries) in the map in two ways: each red triangle has 3 sides, so the number of edges is a multiple of 3, say, $3n$. Each blue triangle has 3 sides, and the infinite region has 5 edges, so the number of edges is a multiple of 3 plus 5, say, $3m + 5$. Thus we have $3n = 3m + 5$. But this is impossible.