Math 145 **Fall 2003**

Homework 1 - Solutions

Dirichlet's box principle

1. **Prove that of 12 distinct two-digin numbers, we can select two with a two-digit difference of the form aa.**

   When we divide a number by 11, there are 11 possible remainders. Since we have 12 numbers (and thus more numbers than possible remainders), by Dirichlet's principle, at least two numbers have the same remainder. Their difference is then divisible by 11. Every two-digin number that is divisible by 11, has the form aa (such numbers are 11, 22, 33, ..., 99).

2. **Three hundred points are selected inside a cube with edge 7. Prove that we can place a small cube with edge 1 inside the big cube such that the interior of the small cube does not contain any of the selected points.**

   Divide the cube into 343 small cubes with edge 1 each. Each point is inside at most one small cube (if a point is on the boundary of a small cube, then it is not inside any small cube). Since there are more small cubes than points, there is a small cube (actually, there are at least 43 of them) that doesn't contain any points inside it.

3. **Let a_1, a_2, a_3, and a_4 be integers. Show that the product $\prod_{1\le i<j\le 4}(a_i - a_j)$ is divisible by 12.**

   First we will show that the product

   $$\prod_{1\le i<j\le 4}(a_i - a_j) = (a_1 - a_2)(a_1 - a_3)(a_1 - a_4)(a_2 - a_3)(a_2 - a_4)(a_3 - a_4)$$

   is divisible by 3. When we divide a number by 3, there are 3 possible remainders. Since we have 4 numbers, there are at least two numbers with the same remainder. Their difference is divisible by 3.

   Now we will show that the product is divisible by 4. More precisely, we will show that at least 2 out of 6 multiples in the product are even. Each number a_i is either odd or even. So think of 2 boxes, one box for even numbers, and one for odd

numbers. Consider the following two cases:
I. There are 2 numbers in each box, i.e. there are 2 even and 2 odd numbers. Then the difference of the even numbers is even, and the difference of the odd numbers is even.
II. There are at least 3 numbers in one box. Then their pairwise differences (at least 3 differences) are even.
Since at least 2 differences are even, and the product of two even numbers is divisible by 4, the whole product is divisible by 4.

So we have that the given product is divisible by both 3 and 4. Therefore it is divisible by 12.

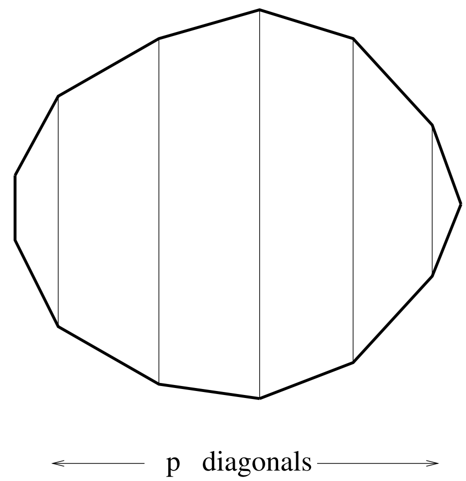
4. **Prove that in any convex $2n$-gon, there is a diagonal not parallel to any side.**

"Make" a box for each side. There will be $2n$ boxes. We will "put" a diagonal into a box if it is parallel to the corresponding side.

box 1: diagonals parallel to side 1	...	box $2n$: diagonals parallel to side $2n$

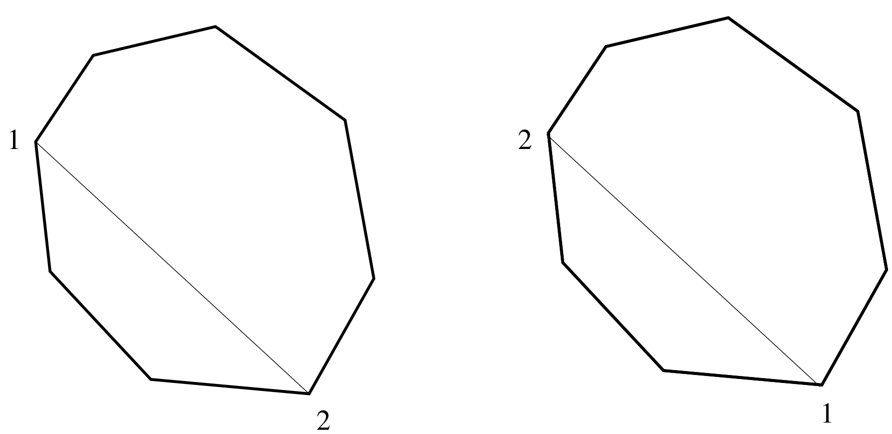
We will figure out the maximal possible number of diagonals that can be parallel to one side (and thus parallel among themselves), i.e. the maximal possible number of diagonals in each box, and we will figure out how many diagonals we have in a $2n$-gon. We will show that $2n$ times the maximal number of diagonals in each box is less than the number of diagonals in a $2n$-gon, thus there is not enough space for all the diagonals in our boxes. Therefore, there is a diagonal that is not in any box, and thus not parallel to any side.

Let p be the maximal possible number of diagonals parallel to the same side. We will find a condition on p. Notice that the vertices of these p diagonals and the 2 vertices of the side they are all parallel to, are distinct (because if 2 line segments have a common vertex, they can not be parallel). Let us draw the $2n$-gon so that all these p diagonals and the parallel side are vertical, with the side on the left. Then we must also have at least one vertex on the right (because the rightmost line segment must be inside the $2n$-gon):



Thus the number of vertices in this figure is at least $2p+2+1$. We have $2n \geq 2p+3$. Since $2n$ is an even number and $2p+3$ is odd, we must have $2n \geq 2p+4$. Then $2n-4 \geq 2p$, and $n-2 \geq p$. So there may be at most $n-2$ diagonals in the same box.

Now, there are $\dfrac{2n(2n-3)}{2} = n(2n-3)$ diagonals in a $2n$-gon because for every diagonal, there are $2n$ ways to choose the first vertex. Once the first vertex has been chosen, there are $2n-3$ ways to choose the second vertex (because the first vertex and its immediate neighbours can not be chosen as the second vertex). But this way we counted each diagonal twice:

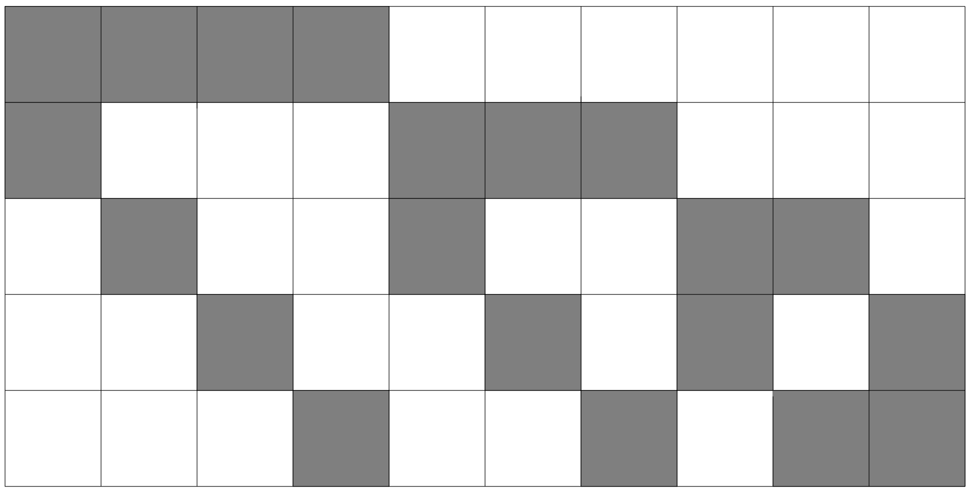


So we divide by 2.

Thus we have $2n$ boxes, at most $n-2$ diagonals may be in the same box, therefore at most $2n(n-2) = 2n^2-4$ diagonals may be in the boxes. But we have $n(2n-3) = 2n^2-3n$ diagonals. Since $2n^2-3n > 2n^2-4$, there is a diagonal which is not in any box, and thus not parallel to any side.

5. **Using 4 colors, we color a 5×41 block checkerboard. Prove that, whichever way we color the blocks, there exist at least one same-color-corner rectangle.**

   Since we have 5 rows and only 4 colors, every column has some color repeated. Since there are 41 columns, there is a color that is repeated (at least twice) in at least 11 columns. So each of these 11 columns contains at least 2 blocks of that color. Choose any 2. There are 10 ways to choose 2 blocks out of 5:



   Since there are at least 11 columns with that color repeated, there are at least 2 columns that have the same 2 blocks of that color. Then we have a same-color-corner-rectangle: