Homework 1

Dirichlet's box principle

Do these by 5 September 2003, 5 points each:

1. Prove that of 12 distinct two-digit numbers, we can select two with a two-digit difference of the form aa.
2. Three hundred points are selected inside a cube with edge 7. Prove that we can place a small cube with edge 1 inside the big cube such that the interior of the small cube does not contain any of the selected points.
3. Let a_1, a_2, a_3, and a_4 be integers. Show that the product $\prod_{1 \leq i < j \leq 4} (a_i - a_j)$ is divisible by 12.
4. Prove that in any convex $2n$-gon, there is a diagonal not parallel to any side.
5. Using 4 colors, we color a 5×41 block checkerboard. Prove that, whichever way we color the blocks, there exist at least one same-color-corner rectangle.

For extra credit, you may work in groups and submit it any time during the semester: The number theory fact given below was needed in my Ph.D. thesis (which is in the area of homological algebra and algebraic topology). My proofs used Dirichlet's principle.

1. (hard) If p is and odd prime, then any element of $\mathbb{Z}_{p^{k+1}}$ of the form $1 + p^2 a$ is the p-th power of some element of the form $1 + pb$.

   Hint 1: If p is odd, $\mathbb{Z}^*_{p^{k+1}}$ is cyclic. What is its order?

   Hint 2: You might want to use the following lemma. Given natural numbers c, d, and f, the equation $cx = d \pmod{f}$ has at most c solutions mod f. In fact, if c divides f, then this equation has either c or 0 solutions. What if c does not divide f? When does the equation has solutions (find a necessary and sufficient condition on c, d, and f).
2. (even harder) There are analogous statements for $p = 2$. In fact, there are closely related facts that are well-known, but I do not know their proofs. Let me know if you are interested in doing a project on this.