Math 145 **Fall 2003**

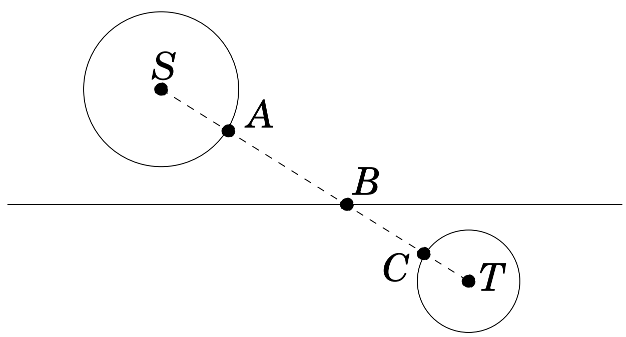
Homework 10 - Solutions

Symmetry, etc.

In all the problems below, "find" means "construct", or "draw". You do not have to calculate the locations of all the points. Assume that solutions exist.

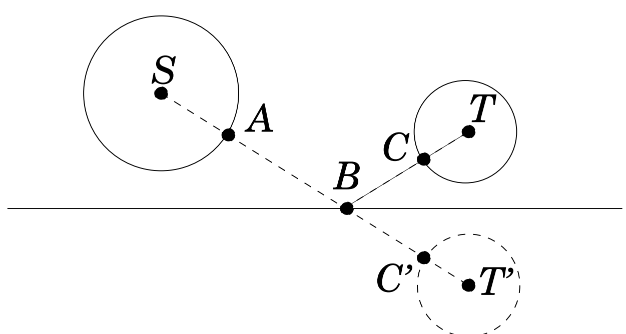
1. **Two circles and a line are given. Suppose that none of them intersect. Find a point A on the first circle, a point B on the line, and a point C on the second circle such that $AB + BC$ is a minimum.**

   Case 1: the circles lie on the opposite sides of the line.



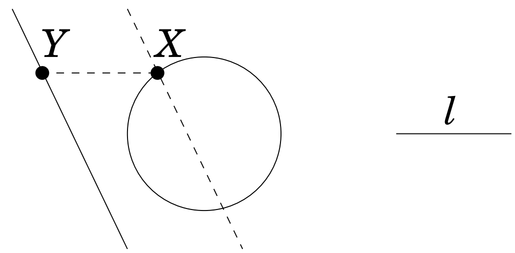
   Let S and T be the centers of the given circles, and let r_1 and r_2 be their radii. Draw a line through the centers of the circles (let's call it L). Let A be the intersection point (closest to the given line) of L and the first circle, let B be the intersection point of L and the given line, and let C be the intersection point (closest to the given line) of L and the second circle. This choice of A, B, and C minimizes $SA + AB + BC + CT$ because the shortest path from S to T is the straight line. Since $SA = r_1$ and $CT = r_2$ are fixed, this choice of A, B, and C minimizes $AB + BC$.

   Case 2: the circles lie on the same side of the line.



   Reflect the second circle about the given line, and draw a straight line through the center of the first circle and the center of the new one. Let the intersection points A, B, and C' be as above. Reflect the point C' about the given line, get a point C on the second (old) circle. This choice of A, B, and C minimizes $AB+BC$ because $AB+BC = AB+BC'$, and we have seen above that this choice of points minimized $AB + BC'$.

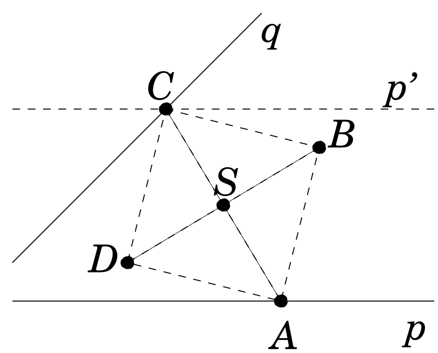
2. **A circle, a line, and a distance l are given. Find a point X on the circle, and a point Y on the line, such that the segment XY is horizontal and has length l.**



Move the line a distance l horizontally (to the right or to the left depending on where the circle is). Let X be an intersection point of the new line and the circle. Move X back, get the point Y on the old line. Then XY is horizontal and has length l. (Note: if the new line and the circle do not intersect, then there is no solution.)

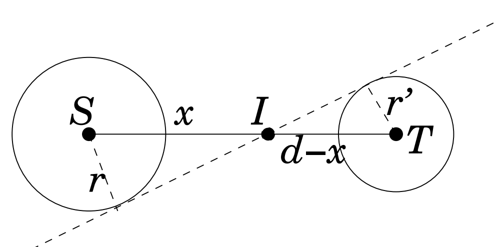
3. **Two distinct lines p and q are given, and a point S. Draw a square $ABCD$ that satisfies the following conditions:**

    - **Point S is the center of the square.**
    - **The vertex A of the square lies on the line p.**
    - **The vertex C, the opposite of the vertex of A, lies on the line q.**



Rotate the line p through an angle of 180 degrees around S. Let's call the new line p'. Let C be the intersetion point of p' and q. (Note: if p' and q do not intersect, then there is no solution.) Rotate C back - get a point A on the old line p. Thus we have that A, S, and C lie on the same line, and $SA = SC$. Now to find the remaining vertices of the square, rotate A through 90 degrees around S (in both directions).

4. **Two circles are given. Draw a line that is tangent to both circles and such that the circles lie on opposite sides of the line.**

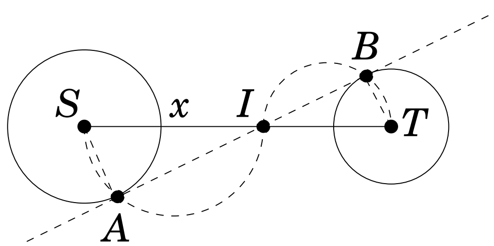


Let the centers of the circles be S and T, their radii r and r', and the distance between their centers d. Draw a line through the centers of the circles. We know that it must cross the common tangent line that we are looking for. Let's find the location of the intersection point I. Let the distance between the center of the first circle and the intersetion point be x, then the distance between the center of the second circle and the intersection point is $d - x$.

From similar triangles we see that $\frac{x}{r} = \frac{d-x}{r'}$.

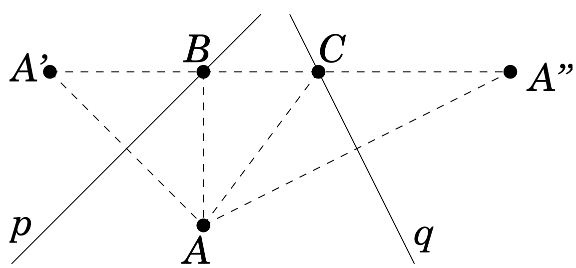
Solving this equation for x gives $x = \frac{r'd}{r+r'}$.

Once we have this intersection point, we draw semicircles with diameters x and $d-x$, and find the intersection points A and B of these semicircles with the given circles. These are the points where the tangent line touches the circles, so we draw a line through these points.



(We know that an angle inscribed in a semicircle is 90 degrees, so both SA and AI are perpendicular, and TB and BI are perpendicular, thus AB is the common tangent.)

5. **A point A and two lines, p and q, are given. Find a point B on the line p, and a point C on the line q, such that the perimeter of the triangle ABC is a minimum.**



Reflect the point A about p, get a point A'. Reflect A about q, get a point A''. We want to find B and C on p and q respectively so that to minimize the perimeter of the triangle ABC. Since $AB + BC + CA = A'B + BC + CA''$, the problem is equivalent to minimizing $A'B + BC + CA''$. But this is minimimized when A', B, C, and A'' line on one line (because the shortest path from A' to A'' is the straight line). Thus we connect A' and A'', and let B and C be the intersection points of $A'A''$ and the lines p and q respectively.