Math 145 **Fall 2003**

Homework 11 - Solutions

Working backwards

1. **Let $a = 96$ and $b = 54$. Find the greatest common divisor d of a and b, and use Euclid's algorithm to find x and y such that $xa + yb = d$.**

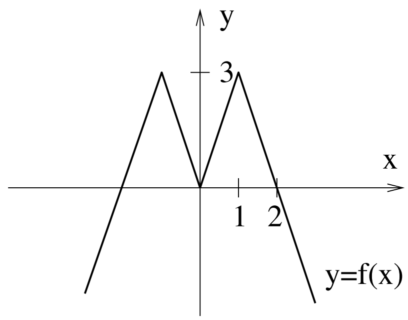
   $96 = 1 \cdot 54 + 42$
   $54 = 1 \cdot 42 + 12$
   $42 = 3 \cdot 12 + 6$
   $12 = 2 \cdot 6$
   Thus $d = (96, 54) = 6$.
   $6 = 42 - 3 \cdot 12$
   $= 42 - 3(54 - 1 \cdot 42) = 4 \cdot 42 - 3 \cdot 54$
   $= 4(96 - 1 \cdot 54) - 3 \cdot 54 = 4 \cdot 96 - 7 \cdot 54, \{mboxsox = 4$ and $y = -7$.

2. **Find a and b such that in Euclid's algorithm $r_7 = (a, b)$. Write out all the divisions.**

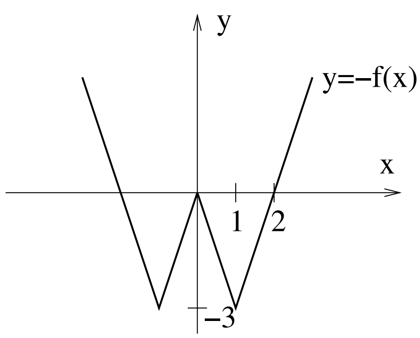
   We want to find a and b such that
   $a = q_1 \cdot b + r_1$
   $b = q_2 \cdot r_1 + r_2$
   $r_1 = q_3 \cdot r_2 + r_3$
   $r_2 = q_4 \cdot r_3 + r_4$
   $r_3 = q_5 \cdot r_4 + r_5$
   $r_4 = q_6 \cdot r_5 + r_6$
   $r_5 = q_7 \cdot r_6 + r_7$
   $r_6 = q_8 \cdot r_7$
   Choose any numbers for r_7 and all the quotients q_i, and work backwards to find all the numbers r_i, b, and a. For example,

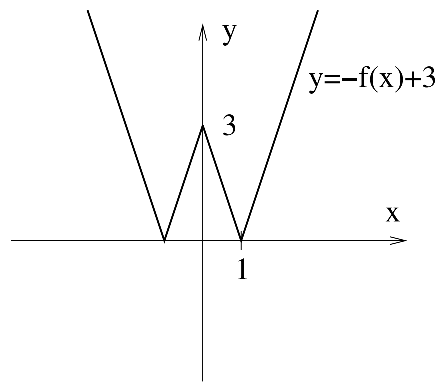
$a = q_1 \cdot b + r_1$	...	$220 = 1 \cdot 127 + 93$
$b = q_2 \cdot r_1 + r_2$	...	$127 = 1 \cdot 93 + 34$
$r_1 = q_3 \cdot r_2 + r_3$	...	$93 = 2 \cdot 34 + 25$
$r_2 = q_4 \cdot r_3 + r_4$	$\uparrow$	$34 = 1 \cdot 25 + 9$
$r_3 = q_5 \cdot r_4 + r_5$	$25 = 2 \cdot 9 + 7$	$25 = 2 \cdot 9 + 7$
$r_4 = q_6 \cdot r_5 + r_6$	$9 = 1 \cdot 7 + 2$	$9 = 1 \cdot 7 + 2$
$r_5 = q_7 \cdot r_6 + r_7$	$7 = 3 \cdot 2 + 1$	$7 = 3 \cdot 2 + 1$
$r_6 = q_8 \cdot r_7$	$2 = 2 \cdot 1$	$2 = 2 \cdot 1$

3. **Find a formula for the function whose graph is shown below.**

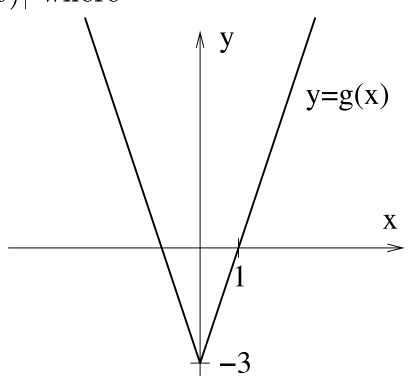


Reflect the given graph about the x-axis (i.e. multiply the function by -1) and shift 3 units upward (i.e. add 3).





Then $-f(x)+3 = |g(x)|$ where



Shifting the graph of $g(x)$ 3 units upward will give the graph of $|3x|$, therefore
$g(x)+3 = |3x|$
$g(x) = |3x| - 3$
$-f(x)+3 = ||3x|-3|$
$f(x) = 3 - ||3x|-3|$

4. **Suppose you are writing a calculus book. You want to find a few cubic polynomials $f(x) = ax^3 + bx^2 + cx + d$ (preferably with integer coefficients) whose critical numbers are integers. (Recall that a critical number is a value of x at which the derivative is equal to 0.) How would you find such polynomials? Use your strategy to find a couple of polynomials.**

   The derivative of a cubic polynomial is a quadratic polynomial. We want that quadratic polynomial to have integer roots. Instead of trying random coefficients a, b, c, and d, let's choose the roots of the quadratic polynomial (the derivative of f), and then find f:
   Choose the roots, e.g. $r_1 = 3$ and $r_2 = 5$.
   $(x-3)(x-5) = x^2 - 8x + 15$.
   Now $f(x)$ can be any antiderivative of this polynomial, say, $\frac{1}{3}x^3 - 4x^2 + 15x - 3$.
   However, we want it to have integer coefficients, so let's mulitply this function by 3:
   $f(x) = x^3 - 12x^2 + 45x - 9$. (Then $f'(x) = 3x^2 - 24x + 45 = 3(x^2 - 8x + 15) = 3(x-3)(x-5)$ has integer roots.)
   Here is another choice of roots and the constant d:
   $r_1 = -3$, $r_2 = 4$, $(x+3)(x-4) = x^2 - x - 12$, an antiderivative is $\frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x - \frac{1}{6}$, multiply by 6:
   $f(x) = 2x^3 - 3x^2 - 72x - 1$. (Then $f'(x) = 6x^2 - 6x - 72 = 6(x^2 - x - 12) = 6(x+3)(x-4)$ has integer roots.)

5. **Two players play the following game.**

   - **Turns alternate.**
   - **At each turn, a player removes 1, 2, 3, or 4 counters from a pile that had initially 27 counters.**
   - **The game ends when all counters have been removed.**
   - **The player who takes the last counter loses.**

   Find a winning strategy for one of the players.

   We want to force our opponent to take the last counter. Thus we have to leave 1 counter on our last turn. To ensure that we'll be able to do that, we'll leave 6 counters on our next to last turn (then if our opponent takes 1, we take 4 and leave 1; if our opponent takes 2, we take 3; if they take 3, we take 2; if they take 4, we take 1). On the turn before the next to last we'll leave 11... and so on. Thus we have to go first, take 1 counter and leave 26. Then no matter how our opponent plays we'll be able to leave 21, 16, 11, 6, 1.