Homework 11 - Solutions

Working backwards

1. Let $a = 96$ and $b = 54$. Find the greatest common divisor $d$ of $a$ and $b$, and use Euclid’s algorithm to find $x$ and $y$ such that $xa + yb = d$.

\[
\begin{align*}
96 &= 1 \cdot 54 + 42 \\
54 &= 1 \cdot 42 + 12 \\
42 &= 3 \cdot 12 + 6 \\
12 &= 2 \cdot 6
\end{align*}
\]

Thus $d = (96, 54) = 6$.

\[
\begin{align*}
6 &= 42 - 3 \cdot 12 \\
&= 42 - 3(54 - 1 \cdot 42) = 4 \cdot 42 - 3 \cdot 54 \\
&= 4(96 - 1 \cdot 54) - 3 \cdot 54 = 4 \cdot 96 - 7 \cdot 54,
\end{align*}
\]

\[\{mbox{sox} = 4 \text{ and } y = -7.\]

2. Find $a$ and $b$ such that in Euclid’s algorithm $r_7 = (a, b)$. Write out all the divisions.

We want to find $a$ and $b$ such that

\[
\begin{align*}
a &= q_1 \cdot b + r_1 \\
b &= q_2 \cdot r_1 + r_2 \\
r_1 &= q_3 \cdot r_2 + r_3 \\
r_2 &= q_4 \cdot r_3 + r_4 \\
r_3 &= q_5 \cdot r_4 + r_5 \\
r_4 &= q_6 \cdot r_5 + r_6 \\
r_5 &= q_7 \cdot r_6 + r_7 \\
r_6 &= q_8 \cdot r_7
\end{align*}
\]

Choose any numbers for $r_7$ and all the quotients $q_i$, and work backwards to find all the numbers $r_i$, $b$, and $a$. For example,

\[
\begin{align*}
a &= q_1 \cdot b + r_1 & 220 &= 1 \cdot 127 + 93 \\
b &= q_2 \cdot r_1 + r_2 & 127 &= 1 \cdot 93 + 34 \\
r_1 &= q_3 \cdot r_2 + r_3 & 93 &= 2 \cdot 34 + 25 \\
r_2 &= q_4 \cdot r_3 + r_4 & 34 &= 1 \cdot 25 + 9 \\
r_3 &= q_5 \cdot r_4 + r_5 & 25 &= 2 \cdot 9 + 7 \\
r_4 &= q_6 \cdot r_5 + r_6 & 25 &= 2 \cdot 9 + 7 \\
r_5 &= q_7 \cdot r_6 + r_7 & 9 &= 1 \cdot 7 + 2 \\
r_6 &= q_8 \cdot r_7 & 9 &= 1 \cdot 7 + 2
\end{align*}
\]
3. Find a formula for the function whose graph is shown below.

Reflect the given graph about the x-axis (i.e. multiply the function by $-1$) and shift 3 units upward (i.e. add 3).

Then $-f(x) + 3 = |g(x)|$ where

Shifting the graph of $g(x)$ 3 units upward will give the graph of $|3x|$, therefore

$g(x) + 3 = |3x|

\begin{align*}
g(x) &= |3x| - 3 \\
-f(x) + 3 &= ||3x| - 3|
\end{align*}

\begin{align*}
f(x) &= 3 - ||3x| - 3|
\end{align*}
4. Suppose you are writing a calculus book. You want to find a few cubic polynomials \( f(x) = ax^3 + bx^2 + cx + d \) (preferably with integer coefficients) whose critical numbers are integers. (Recall that a critical number is a value of \( x \) at which the derivative is equal to 0.) How would you find such polynomials? Use your strategy to find a couple of polynomials.

The derivative of a cubic polynomial is a quadratic polynomial. We want that quadratic polynomial to have integer roots. Instead of trying random coefficients \( a, b, c, \) and \( d \), let’s choose the roots of the quadratic polynomial (the derivative of \( f \)), and then find \( f \):

Choose the roots, e.g. \( r_1 = 3 \) and \( r_2 = 5 \).
\[
(x - 3)(x - 5) = x^2 - 8x + 15.
\]

Now \( f(x) \) can be any antiderivative of this polynomial, say, \( \frac{1}{3}x^3 - 4x^2 + 15x - 3 \). However, we want it to have integer coefficients, so let’s multiply this function by 3:
\[
f(x) = x^3 - 12x^2 + 45x - 9. \quad \text{(Then } f'(x) = 3x^2 - 24x + 45 = 3(x^2 - 8x + 15) = 3(x - 3)(x - 5) \text{ has integer roots.)}
\]

Here is another choice of roots and the constant \( d \):
\[
r_1 = -3, r_2 = 4, \quad (x + 3)(x - 4) = x^2 - x - 12,
\]

an antiderivative is \( \frac{1}{3}x^3 - \frac{1}{2}x^2 - 12x - \frac{1}{6} \), multiply by 6:
\[
f(x) = 2x^3 - 3x^2 - 72x - 1. \quad \text{(Then } f'(x) = 6x^2 - 6x - 72 = 6(x^2 - x - 12) = 6(x + 3)(x - 4) \text{ has integer roots.)}
\]

5. Two players play the following game.

- Turns alternate.
- At each turn, a player removes 1, 2, 3, or 4 counters from a pile that had initially 27 counters.
- The game ends when all counters have been removed.
- The player who takes the last counter loses.

Find a winning strategy for one of the players.

We want to force our opponent to take the last counter. Thus we have to leave 1 counter on our last turn. To ensure that we’ll be able to do that, we’ll leave 6 counters on our next to last turn (then if our opponent takes 1, we take 4 and leave 1; if our opponent takes 2, we take 3; if they take 3, we take 2; if they take 4, we take 1). On the turn before the next to last we’ll leave 11... and so on. Thus we have to go first, take 1 counter and leave 26. Then no matter how our opponent plays we’ll be able to leave 21, 16, 11, 6, 1.