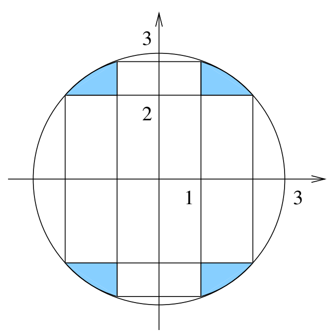
Math 145 **Fall 2003**

Homework 12 - Solutions

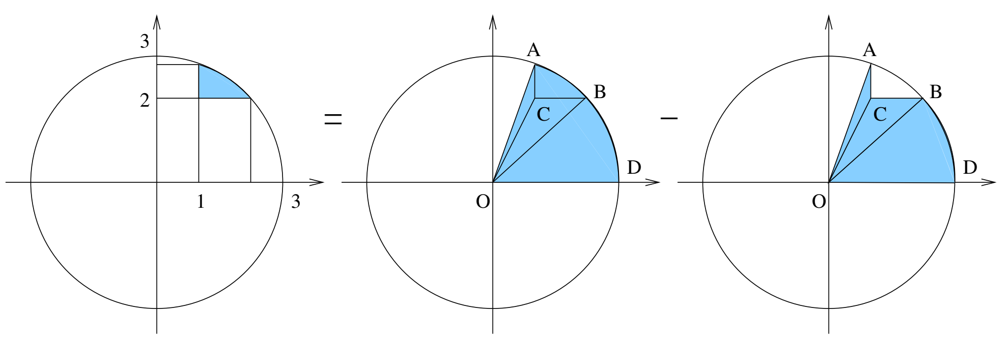
Calculus

1. **Find the n-th derivative of $f(x) = \dfrac{1}{x^2+x}$.**
First find the partial fraction decomposition, i.e. A abd B such that
$\dfrac{1}{x^2+x} = \dfrac{A}{x} + \dfrac{B}{x+1}$.
Multiply both sides by $x^2+x = x(x+1)$:
$1 = A(x+1) + Bx$
$1 = (A+B)x + A \qquad \Rightarrow \qquad A+B = 0$ and $A = 1$, then $B = -1$.
Thus $f(x) = \dfrac{1}{x^2+x} = \dfrac{1}{x} - \dfrac{1}{x+1} = x^{-1} - (x+1)^{-1}$.
$f'(x) = -x^{-2} + (x+1)^{-2}$
$f''(x) = 2x^{-3} - 2(x+1)^{-3}$
$f'''(x) = -2 \cdot 3x^{-4} + 2 \cdot 3(x+1)^{-4}$
...
$f^{(n)}(x) = (-1)^n n! x^{-n-1} - (-1)^n n!(x+1)^{-n-1}$
Note: this formula can be proved by Mathematical Induction.

2. **Sketch the region $S = \{(x,y) \mid |x| \geq 1,\ |y| \geq 2,\ x^2+y^2 \leq 9\}$ and find its area.**
Our region consists of 4 parts of equal area.



The total area of our region is 4 times the area of each part.



The area of each part is the area of sector OAD minus the area of sector OBD minus the area of triangle OBC minus the area of triangle OAC.
A has coordinates $(1, \sqrt{8})$, thus $\text{Area}_{OAD} = \dfrac{9\arccos(1/3)}{2}$.
B has coordinates $(\sqrt{5}, 2)$, thus $\text{Area}_{OBD} = \dfrac{9\arcsin(2/3)}{2}$.
OBC has base $BC = \sqrt{5} - 1$ and height $h_{BC} = 2$, thus $\text{Area}_{OBC} = \dfrac{2(\sqrt{5}-1)}{2}$.

OAC has base $AC = \sqrt{8} - 2$ and height $h_{AC} = 1$, thus $\text{Area}_{OAC} = \dfrac{\sqrt{8}-2}{2}$.

Then $\text{Area}_{ABC} = \dfrac{9\arccos(1/3) - 9\arcsin(2/3) - 2(\sqrt{5}-1) - (\sqrt{8}-2)}{2}$

$= \dfrac{9\arccos(1/3) - 9\arcsin(2/3) - 2\sqrt{5} - \sqrt{8} + 4}{2}$

The total area is then $2(9\arccos(1/3) - 9\arcsin(2/3) - 2\sqrt{5} - \sqrt{8} + 4)$

3. **Find a such that the area of the region bounded by the line $y = ax$ and the parabola $y = x^2$ is equal to 1.**

To find the intersection points of the line $y = ax$ and the parabola $y = x^2$, solve $ax = x^2$. The roots are $x = 0$ and $x = a$, thus the intersection points are $(0,0)$ and (a, a^2).

If $a > 0$, the area is $\displaystyle\int_0^a (ax - x^2)dx = \left(a\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 = \frac{a^3}{2} - \frac{a^3}{3} = \frac{a^3}{6}$.

We want the area to be equal to 1, so $\dfrac{a^3}{6} = 1 \quad \Rightarrow \quad a^3 = 6 \quad \Rightarrow \quad a = \sqrt[3]{6}$

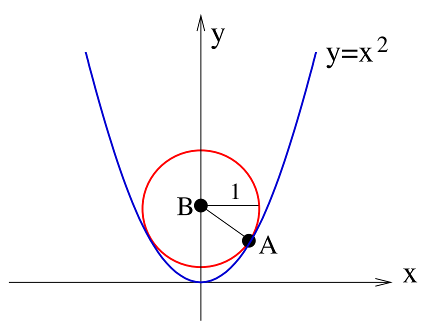
If $a < 0$, then the area is $\displaystyle\int_a^0 (ax - x^2)dx = -\frac{a^3}{6} \quad \Rightarrow \quad a = -\sqrt[3]{6}$.

4. **Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \ldots$**

$\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \frac{1}{2^7} + \ldots = \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots\right)$

$\displaystyle= \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots\right) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{4}} = \frac{2}{3}$

5. **The figure below shows a circle with radius 1 inscribed in the parabola $y = x^2$. Find the center of the circle.**



Let the point A (with positive x-coordinate) where the circle touches the parabola be (a, a^2), and the center B of the circle be $(0, b)$. Then the distance between these points is 1, thus

$$a^2 + (b - a^2)^2 = 1.$$

The slope of the parabola at the point (a, a^2) is $2a$ (the derivative of x^2 at $x = a$), then the slope of AB is $-\dfrac{1}{2a}$ (since AB and the parabola are orthogonal at (a, a^2)). Thus we have

$$\frac{b - a^2}{0 - a} = -\frac{1}{2a}.$$

The last equation gives $b - a^2 = \dfrac{1}{2}$, then from the first equation we have

$a^2 + \dfrac{1}{4} = 1 \quad \Rightarrow \quad a^2 = \dfrac{3}{4}$. Then $b = a^2 + \dfrac{1}{2} = \dfrac{3}{4} + \dfrac{1}{2} = \dfrac{5}{4}$.