# Homework 2 - Solutions

## Mathematical Induction

1. **Prove the following identity for Fibonacci numbers: $F_{n-1}F_{n+1} = F_n^2 + (-1)^n$.**
   Basis step. Check for $n = 1$: $F_0F_2 = F_1^2 + (-1)^1$. Since $F_0 = 0$, $F_1 = F_2 = 1$, we have $0 \cdot 2 = 1 + (-1)$ which is true.
   Inductive step. Assume the identity holds for $n = k$, i.e.

   $$F_{k-1}F_{k+1} = F_k^2 + (-1)^k.$$

   We want to show that it then holds for $n = k + 1$:

   $$F_{(k+1)-1}F_{(k+1)+1} = F_{k+1}^2 + (-1)^{k+1},$$

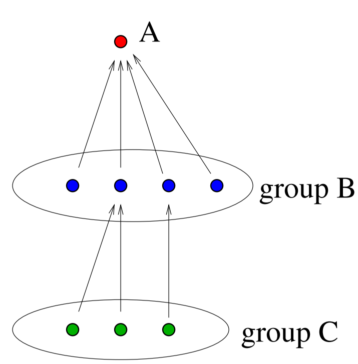
   or, equivalently,

   $$F_kF_{k+2} = F_{k+1}^2 + (-1)^{k+1}.$$

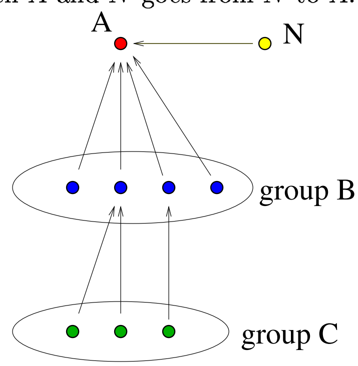
   We have

   $$\begin{aligned} F_kF_{k+2} &= F_k(F_k + F_{k+1}) \\ &= F_k^2 + F_kF_{k+1} \\ &= F_{k-1}F_{k+1} - (-1)^k + F_kF_{k+1} \\ &= F_{k+1}(F_{k-1} + F_k) + (-1) \cdot (-1)^k \\ &= F_{k+1}^2 + (-1)^{k+1}. \end{aligned}$$

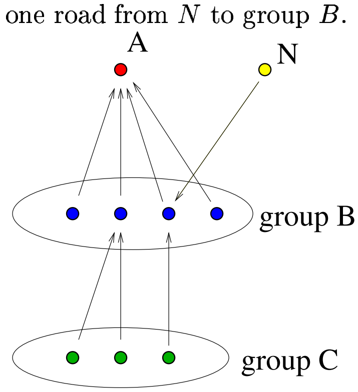
2. **Every road in Sikinia is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every other city either directly or via at most one other city.**
   Basis step. For $n = 1$ city we are done because there is no "any other city". (The step $n = 2$, in which case we have 2 cities and one road between them, so one city can be reached from the other, was also accepted.)
   Inductive step. Assume the statement is true for $n = k$, that is, for any system of roads between $k$ cities, there is a city (let us call it city $A$) that can be reached from any other city either directly or via at most one other city. Let us call those cities from which there are direct roads to $A$ group $B$, and the rest of the cities group $C$. Then from every city in group $C$ there is a road to at least one city in group $B$:



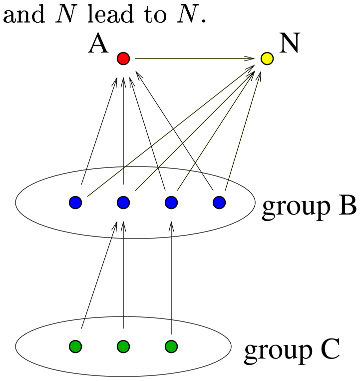
Now we add a $(k+1)$-st city, let us call it N (new). Consider the following 3 cases:
Case I. The road between $A$ and $N$ goes from $N$ to $A$.




Then we put $N$ into the group $B$, and $A$ is still "a solution city".
Case II. There is at lest one road from $N$ to group $B$.



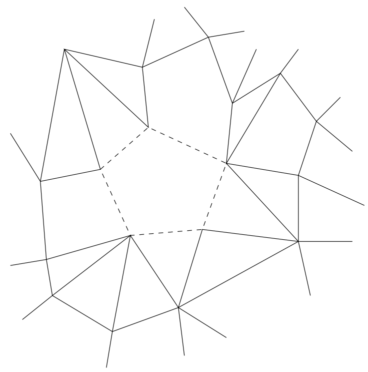

Then we put $N$ into the group $C$, and $A$ is still a solution city.
Case III. None of the above: the road between $A$ and $N$ goes from $A$ to $N$, and all the roads between group $B$ and $N$ lead to $N$.



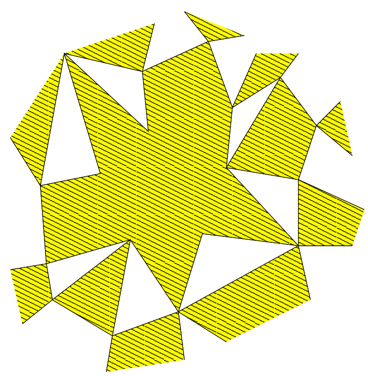

Then $N$ is a new solution city, and $A$ will join group $B$.

3. **A map can be properly colored (see problems done in class for definition of a proper coloring) with two colors iff ("iff" means "if and only if") all of its vertices have even degree.**
   First of all, if there is at least one vertex has odd degree, than there is an odd number of regions around it, and it is obvious that they can not be properly colored with two colors.
   We will show that if the degree of each vertex is even, then the map can be properly colored with two colors. The induction will be on the number of boundary lines (or curves), and we will use a generalized version of Math induction.
   Basis step. For $n = 0$ boundaries, the whole plane is one big region. We can color it

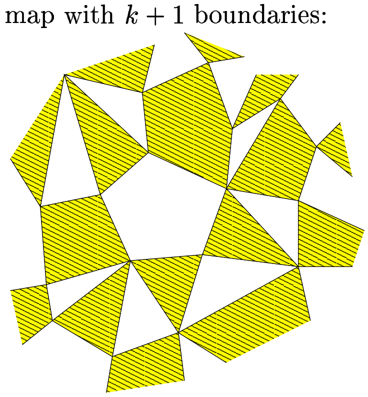
with any color we like.
Inductive step. Suppose any map with less than or equal to $k$ boundaries can be properly colored with two colors. We wish to show that any map with $k+1$ boundaries can be properly colored. Suppose we are given such a map. Remove temporarily all the boundaries of one region (any one; the fact that the degree of each vertex is even guarantees that the boundary of each region is a simple closed curve as shown on the picture, that is, there are no "hanging" edges).



We get a map with less than or equal to $k$ boundaries, and the degree of each vertex is still even. By the inductive assumption this new map can be properly colored. Consider a coloring,



put the boundaries of our region back, and change the color inside it. We get a proper coloring for our original map with $k+1$ boundaries:



4. **Prove that** $1 < \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n+1} < 2$**.**

First we will try to estimate the sum by estimating each term. We see that

$$\frac{1}{3n+1} \leq \text{each term} \leq \frac{1}{n+1},$$

and we have $2n+1$ terms, therefore we have

$$\frac{2n+1}{3n+1} \leq \text{sum} \leq \frac{2n+1}{n+1}.$$

The left inequality doesn't help us, but from the right one we have

$$\text{sum} \leq \frac{2n+1}{n+1} < \frac{2n+2}{n+1} = 2,$$

thus we do not need Math induction for this part. To show that the sum is bigger than 1, we will use Math induction.

Basis step. Check for $n = 1$: $1 < \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$. We calculate $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{13}{12}$, and we see that this is bigger than 1.

Inductive step. Assume the inequality holds for $n = k$, i.e.

$$1 < \frac{1}{k+1} + \frac{1}{k+2} + \ldots + \frac{1}{3k+1}. \tag{1}$$

We want to prove that it holds for $n = k+1$:

$$1 < \frac{1}{(k+1)+1} + \frac{1}{(k+1)+2} + \ldots + \frac{1}{3(k+1)+1},$$

or

$$1 < \frac{1}{k+2} + \frac{1}{k+3} + \ldots + \frac{1}{3k+1} + \frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}. \tag{2}$$

Compare (1) and (2), and notice that we lost the term $\frac{1}{k+1}$ but gained 3 terms $\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4}$. If we can show that we gained more than we lost, then the new sum (for $k+1$) is bigger than 1. Thus we want to show that

$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > \frac{1}{k+1}.$$

The following inequalities are equivalent:

$$\frac{1}{3k+2} + \frac{1}{3k+3} + \frac{1}{3k+4} > \frac{3}{3k+3}$$

$$\frac{1}{3k+2} + \frac{1}{3k+4} > \frac{2}{3k+3}$$

$$\frac{6k+6}{(3k+2)(3k+4)} > \frac{2}{3k+3}$$

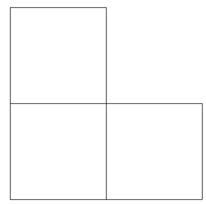
$$\frac{3k+3}{(3k+2)(3k+4)} > \frac{1}{3k+3}$$

$$(3k+3)^2 > (3k+2)(3k+4)$$

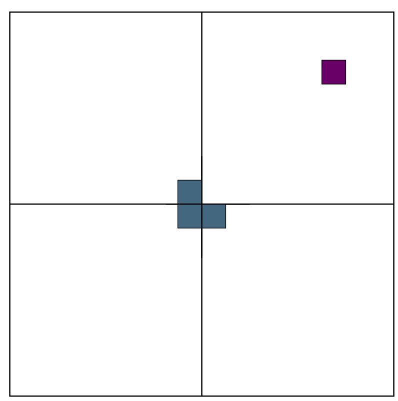
$$9k^2 + 18k + 9 > 9k^2 + 18k + 8,$$

and the last one is obviously true.

5. **If one square of a $2^n \times 2^n$ chessboard is removed, then the remaining board can be covered by L-trominoes, i.e. the figures consisting of 3 squares as shown below.**



(**You can choose which square you want to remove.**)
We will prove a stronger statement: no matter which square is removed, we can cover the rest of the board by L-trominoes.
Basis step. A $2 \times 2$ board with one square removed has the shape of an L-tromino, and thus can be covered by one L-tromino.
Inductive step. Assume that a $2^k \times 2^k$ board with any square removed can be covered by L-trominoes. Now suppose we are given a $2^{k+1} \times 2^{k+1}$ board with one square removed. Divide this board into four $2^k \times 2^k$ boards. One of them has one square removed, and the three others are whole boards. Temporarily remove corner squares from those three whole boards as shown on the picture below.



By the induction assumption, every of these four boards can be covered by L-trominoes. Now place one more L-tromino in the center to cover the 3 squares that we temporarily removed. We are done.