Math 145 **Fall 2003**

Homework 3 - Solutions

Logic and types of proofs

1. **Show that $p \to q$ and $\neg q \to \neg p$ are logically equivalent.**

   Construct the truth table:

p	q	$p \to q$	$\neg q$	$\neg p$	$\neg q \to \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

   The columns for $p \to q$ and $\neg q \to \neg p$ are the same, thus the propositions are logically equivalent.

2. **Let $Q(x, y)$ be the statement "$x + y = x - y$", and the domain for both variables is the set of integers. Find the truth values of the following statements. Explain.**

   (a) $Q(2, 0)$
   True because $2 + 0 = 2 - 0$ is true.

   (b) $\forall y Q(1, y)$
   False because "for every y, $1 + y = 1 - y$ is false: for example, if $y = 1$, then $2 \neq 0$.

   (c) $\forall x \exists y Q(x, y)$
   True because for any x, take $y = 0$, then $x + 0 = x - 0$ is true.

   (d) $\forall y \exists x Q(x, y)$
   False because for example if $y = 1$, there is no x such that $x + 1 = x - 1$.

   (e) $\exists y \forall x Q(x, y)$
   True because if $y = 0$, then for any x we have $x + 0 = x - 0$.

   Note: statements (c) and (e) are not equivalent a priori! (c) says that for any x we can find a y such that $Q(x, y)$ is true. It is possible that we will find different values of y for different values of x. While (e) says that there is a value of y that works for any x.

3. **Prove that if a positive integer is divisible by 8 then it is the difference of two perfect squares. Is your proof direct, by contradiction, or by contrapositive? Is it constructive or nonconstructive?**

   An integer divisible by 8 has the form $8n$.
   $8n = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = (2n + 1)^2 - (2n - 1)^2$.

   This prove is direct and constructive: we gave an explicit example of two perfect squares whose difference is equal to $8n$.

4. **Prove that the equation $x^{101} + x^{51} + x + 1 = 0$ has exactly one real solution. Split this into two statements:**

   (a) **the equation has at least one solution. Is your proof constructive or nonconstructive?**

   Let $f(x) = x^{101} + x^{51} + x + 1$. Then $f(-1) = -2 < 0$ and $f(1) = 4 > 0$. By the intermediate value theorem, $f(x)$ has a root.

   This proof is nonconstructive because we did not construct a root, only proved its existence.

   (b) **the equation can not have two distinct roots. Is your proof direct, by contradiction, or by contrapositive?**

   Suppose $f(x)$ has two distinct roots. Then by the mean value theorem, there is a number c between these roots such that $f'(c) = 0$. But $f'(x) = 101x^{100} + 51x^{50} + 1 > 0$ everywhere. We get a contradiction.

   This is a proof by contradiction.

5. **Every odd number is either of the form $4n + 1$ (if it has remainder 1 when divided by 4) or of the form $4n + 3$ (if it has remainder 3). Prove that if an odd number is a perfect square, then it has the form $4n + 1$. Is your proof direct, by contradiction, or by contrapositive? State the converse. Prove or disprove the converse.**

   If an odd number N is a perfect square, then $N = m^2$ where m is odd. Then m can be written in the form $m = 2k+1$. Then $N = m^2 = (2k+1)^2 = 4k^2+4k+1 = 4(k^2 + k) + 1$, so N is of the form $4n + 1$.

   This proof is direct.

   The converse is "if an odd number has the form $4n+1$, then it is a perfect square". This is false because for example $5 = 4 \cdot 1 + 1$ but 5 is not a perfect square.

   5 is a counterexample.