Math 145 **Fall 2003**

Homework 4 - Solutions

Number theory

1. **Show that $\sqrt[3]{25}$ is irrational.**

   Suppose $\sqrt[3]{25}$ is rational. Then it can be wtitten as an irreducible quotient:
   $\sqrt[3]{25} = \dfrac{m}{n}, \quad m, n \in \mathbb{Z}, \quad (m, n) = 1.$
   $25 = \dfrac{m^3}{n^3}$
   $25n^3 = m^3$
   Now there are several ways to get a contradiction.

   Way 1: From the last equation, $5|m$, so $m = 5a$ for some integer a.
   $25n^3 = (5a)^3$
   $25n^3 = 125a^3$
   $n^3 = 5a^3$
   Now $5|n$. Thus both m and n are divisible by 5, which contradicts the condition $(m, n) = 1$.

   Way 2: If $n = 1$, then $25 = m^3$ which is impossible.
   If $n > 1$, then $n|m$ which contradicts $(m, n) = 1$.

   Way 3: We have $5 \cdot 5 \cdot n^3 = m^3$. Both n and m cab be written as products of primes. Since n and m are cubed, the number of 5's on the left is 2 plus a multiple of 3, and the number of 5's on the right is a multiple of 3. This contradicts the fundamental theorem of arithmetic.

2. **If c is a perfect square, what are the possible values of its last (units) digit? Conclude that a number ending with 3 cannot be a perfect square.**

   First notice that if k is the last digit of m, then the last digit of m^2 is that of k^2 because $m = 10n + k$ for some n, and $m^2 = (10n + k)^2 = 100n^2 + 20nk + k^2 = (10n^2 + 2nk) \cdot 10 + k^2$. So we consider all possible last digits and compute their squares: $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, 4^2 ends with 6, 5^2 ends with 5, 6^2 ends with 6, 7^2 ends with 9, 8^2 ends with 4, and 9^2 ends with 1. Thus the last digit of a perfect square can be 0, 1, 4, 5, 6, or 9. Since 3 is not listed, a number ending with 3 cannot be a perfect square.

3. **Show that 3 divides both a and b iff 3 divides $a^2 + b^2$.**

   The number a can have remainder 0, 1, or 2 mod 3. So can b. Therefore we have 9 cases for the pair $\{a, b\}$. We calculate $a^2 + b^2$ mod 3 in each case:

	$a \equiv 0 \pmod 3$	$a \equiv 1 \pmod 3$	$a \equiv 2 \pmod 3$
$b \equiv 0 \pmod 3$	$a^2 + b^2 \equiv 0 \pmod 3$	$a^2 + b^2 \equiv 1 \pmod 3$	$a^2 + b^2 \equiv 1 \pmod 3$
$b \equiv 1 \pmod 3$	$a^2 + b^2 \equiv 1 \pmod 3$	$a^2 + b^2 \equiv 2 \pmod 3$	$a^2 + b^2 \equiv 2 \pmod 3$
$b \equiv 2 \pmod 3$	$a^2 + b^2 \equiv 1 \pmod 3$	$a^2 + b^2 \equiv 2 \pmod 3$	$a^2 + b^2 \equiv 2 \pmod 3$

We see that $a^2 + b^2 \equiv 0 \pmod 3$ if and only if $a \equiv 0 \pmod 3$ and $b \equiv 0 \pmod 3$.

4. **Recall the following problem done in class: "show that a natural number is divisible by 3 iff the sum of its digits is divisible by 3". Show similarly that a natural number is divisible by 9 iff the sum of its digits is divisible by 9. Derive that if the sum of the digits of a number is 66, then it is not a perfect square.**

   A number $N = \underline{a_n a_{n-1} \ldots a_1 a_0}$ (with digits a_n, a_{n-1}, $\ldots$, a_1, a_0) can be written as

$$N = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + \ldots + a_1 \cdot 10 + a_0 = \sum_{k=0}^{n} a_k \cdot 10^k.$$

   The sum of its digits is

$$S = a_n + a_{n-1} + \ldots + a_1 + a_0 = \sum_{k=0}^{n} a_k.$$

   We have $10 \equiv 1 \pmod 9$
   $10^k \equiv 1 \pmod 9$
   $a_k \cdot 10^k \equiv a_k \pmod 9$
   $\sum_{k=0}^{n} a_k \cdot 10^k \equiv \sum_{k=0}^{n} a_k \pmod 9$
   $N \equiv S \pmod 9$
   Thus n is divisible by 9 if and only if S is divisible by 9.

   If the sum of the digits of a number is 66, then the number is divisible by 3 but not divisible by 9. But if a pefect square is divisible by 3 then it must be divisible by 9. Therefore a number with the digital sum 66 cannot be a perfect square.

5. **Show that if n is not prime, then $2^n - 1$ is not prime.**

   If n is composite, then $n = ab$ for some $1 < a, b < n$. Then

$$2^n - 1 = (2^a)^b - 1^b = (2^a - 1)((2^a)^{b-1} + \ldots + 2^a + 1).$$

   Both multiples are bigger than 1: $2^a - 1 > 2^1 - 1 = 1$, and $(2^a)^{b-1} + \ldots + 2^a + 1 > 1$, so $2^n - 1$ is composite.

   If n is neither prime nor composite (1, 0, or negative), $2^n - 1$ is neither prime nor composite (namely, 1, 0, or noninteger).

   Note: we only consider integer values of n in this homework.

Solutions to **extra credit problems** are not provided because they can be submitted at any time during the semester!