Homework 5 - Solutions

Finding a pattern

1. **Find a formula for the n-th term of the sequence whose first few terms are given.**

   (a) **1, 4, 9, 16, 25, 36, 49, ...** $a_n = n^2$

   (b) **8, 10, 12, 14, 16, 18, ...** $a_n = 6 + 2n$

   (c) **3, 1, -1, -3, -5, -7, ...** $a_n = 5 - 2n$

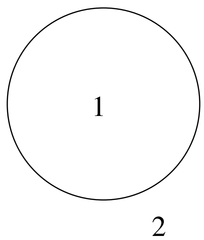
   (d) **1, 2, 1, 4, 1, 6, 1, 8, ...** $a_n = \begin{cases} 1 & \text{if } \ n \text{ is odd} \\ n & \text{if } \ n \text{ is even} \end{cases}$

   (e) **0, 1, 3, 7, 15, 31, ...** $a_n = 2^{n-1} - 1$
   (Notice that if we add 1 to each term, we'll get 1, 2, 4, 8, 16, 32, ... a formula for this sequence is 2^{n-1}. Also, could look at the differences: 1, 2, 4, 8, 16; then $a_n = 1 + 2 + 4 + 8 + ... + 2^{n-2} = 2^{n-1} - 1$.)

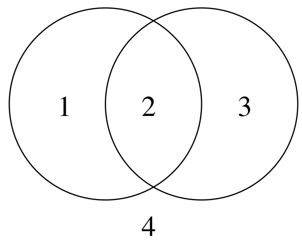
2. **Find the n-th derivative of $f(x) = 2e^{5x}$.**
   $f'(x) = 2 \cdot 5e^{5x}$
   $f''(x) = 2 \cdot 5 \cdot 5e^{5x}$
   $f'''(x) = 2 \cdot 5 \cdot 5 \cdot 5e^{5x}$
   We guess that $f^{(n)}(x) = 2 \cdot 5^n e^{5x}$, and prove this formula by Mathematical Induction.
   Basis step: $f'(x) = 2 \cdot 5e^{5x}$ is true.
   Inductive step: suppose $f^{(k)}(x) = 2 \cdot 5^k e^{5x}$, then $f^{(k+1)}(x) = 2 \cdot 5^k \cdot 5e^{5x} = 2 \cdot 5^{k+1} e^{5x}$.

3. n **circles are given in a plane, such that every pair of circles has 2 intersection points, but no 3 circles have a common point. Into how many regions do they divide the plane?**

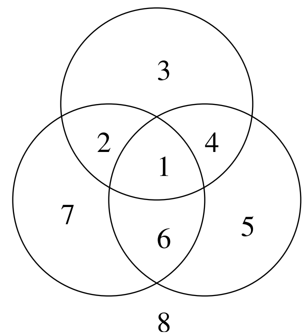
First find the number of regions for some small n:



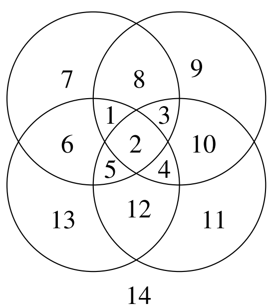
$n = 1$
2 regions



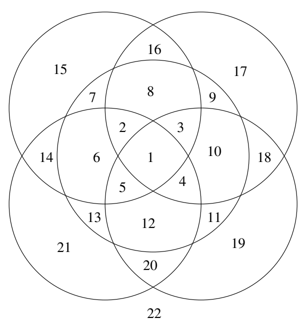
$n = 2$
4 regions



$n = 3$
8 regions



$n = 4$
14 regions



$n = 5$
22 regions

The differences are 2, 4, 6, 8, ... We guess that the differences are all even numbers, so $a_n = 2+2+4+6+...+2(n-1) = 2+2(1+2+3+...+(n-1)) = 2+2\frac{(n-1)n}{2} = 2+(n-1)n = n^2-n+2$.

Now we will prove this formula by Mathematical Induction.

Basis step: If $n = 1$, the formula gives 2, and it is true that there are 2 regions.

Inductive step: Suppose the formula is true for k circles. We add the $(k+1)$-st circle. This new circle intersects the old k circles in $2k$ points. Thus the intersection points divide the new circle into $2k$ arcs. Therefore, the number of regions increases by $2k$ (each arc divides an old region into 2). Then, if k circles divided the plane into $k^2 - k + 2$ regions, $k + 1$ circles will divide it into $k^2 - k + 2 + 2k = k^2 + 2k + 1 - k - 1 + 2 = (k + 1)^2 - (k + 1) + 2$ regions, and the formula holds for $k + 1$.

4. **What is the last digit of 2003^{2003}?**

First notice that the last digit of 2003^n is that of 3^n. So we compute 3^n for small n: 3, 9, 27, 81, 343, ... the last digits 3, 9, 7, and 1 keep repeating. Thus

$$\text{the last digit of } 2003^n = \begin{cases} 3 & \text{if} \quad n \equiv 1 \pmod 4 \\ 9 & \text{if} \quad n \equiv 2 \pmod 4 \\ 7 & \text{if} \quad n \equiv 3 \pmod 4 \\ 1 & \text{if} \quad n \equiv 0 \pmod 4 \end{cases}$$

This can be proved by Mathematical Induction: the basis step is obvious, and the inductive step is as follows. Suppose the statement is true for $n = k$. We want to prove that it is true for $n = k + 1$. Consider 4 cases:

- If $k + 1 \equiv 1 \pmod 4$, then $k \equiv 0 \pmod 4$, thus the last digit of 2003^k is 1. Then the last digit of $2003^{k+1} = 2003^k \cdot 2003$ is $1 \cdot 3 = 3$.
- If $k + 1 \equiv 2 \pmod 4$, then $k \equiv 1 \pmod 4$, thus the last digit of 2003^k is 3. Then the last digit of $2003^{k+1} = 2003^k \cdot 2003$ is $3 \cdot 3 = 9$.
- If $k + 1 \equiv 3 \pmod 4$, then $k \equiv 2 \pmod 4$, thus the last digit of 2003^k is 9. Then the last digit of $2003^{k+1} = 2003^k \cdot 2003$ is that of $9 \cdot 3$ i.e. 7.
- If $k + 1 \equiv 0 \pmod 4$, then $k \equiv 3 \pmod 4$, thus the last digit of 2003^k is 7. Then the last digit of $2003^{k+1} = 2003^k \cdot 2003$ is that of $7 \cdot 3$ i.e. 1.

Since $2003 \equiv 3 \pmod 4$, the last digit of 2003^{2003} is 7.