Math 145 **Fall 2003**

Homework 6 - Solutions

Invariants

1. **Start with the positive integers $1, 2, \ldots, 4n-1$. In each step you may replace any two integers by their difference. Prove that an even integer will be left after $4n-2$ steps.**
When we replace a and b (let $a > b$) by $a-b$, the sum of all the numbers changes by
$$-a - b + (a-b) = -2b \equiv 0 \pmod 2.$$
So the parity of the sum does not change. Initially the sum is
$$1 + 2 + \ldots + (4n-1) = \frac{(4n-1)4n}{2} = (4n-1)2n$$
which is even. Thus the sum of the numbers is always even. Therefore an even number will remain in the end.

2. **Start with the set $\{1, 2, 3, 4\}$. In each step you may add or subtract 2 times one of the numbers to/from another number. Say, you can replace 1 by $1 + 2 \cdot 2$, or by $1 - 2 \cdot 2$, or by $1 + 2 \cdot 3$, etc. Can you reach $\{10, 20, 30, 40\}$?**
When we replace a by $a + 2b$ or $a - 2b$, we do not change its parity (if a is even, then $a \pm 2b$ is even, and if a is odd, then $a \pm 2b$ is odd). Thus the parity of each number will always be the same. Initially we had 2 even and 2 odd numbers. It is not possible to make all of the numbers even.

3. **Each of the numbers 1 to 10^6 is repeatedly replaced by its digital sum until we reach 10^6 one-digit numbers. For example, 987654 is replaced by $9+8+7+6+5+4 = 39$, then 39 is replaced by $3+9 = 12$, and finally, 12 is replaced by $1+2 = 3$. Among these 10^6 one-digit numbers, will we have more 1's or 2's?**
We have seen that any number is conguent to the sum of its digits mod 9. Thus when we replace a number by the sum of its digits, its remainder mod 9 does not change. Thus the question is equivalent to whether there are more numbers congruent to 1 or congruent to 2 mod 9 (among 1, 2, …, 10^6). Remainders mod 9 are 1, 2, 3, …, 8, 0, and they repeat. The last number is $10^6 \equiv 1 \pmod 9$, thus there will be more 1's.

4. **The integers $1, 2, 3, 4, 5, 6$ are arranged in any order on 6 places numbered 1 through 6. Now we add its place number to each integer. Prove that there are two among the sums which have the same remainder mod 6.**
Let the integers be a_1, a_2, a_3, a_4, a_5, and a_6. The sets $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ and $\{0, 1, 2, 3, 4, 5\}$ are equal. Thus the sum of all the a_i's is

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 1 + 2 + \ldots + 6 = 21.$$

We add its place number to each integer and get

$$a_1 + 1,\ a_2 + 2,\ a_3 + 3,\ a_4 + 4,\ a_5 + 5,\ a_6 + 6.$$

Then the sum of these sums is

$$\begin{aligned}&(a_1 + 1) + (a_2 + 2) + (a_3 + 3) + (a_4 + 4) + (a_5 + 5) + (a_6 + 6)\\ &= (a_1 + a_2 + a_3 + a_4 + a_5 + a_6) + (1 + 2 + 3 + 4 + 5 + 6) = 21 + 21 = 42.\end{aligned}$$

If all the sums $a_1 + 1$, $a_2 + 2$, $a_3 + 3$, $a_4 + 4$, $a_5 + 5$, and $a_6 + 6$ have different remainders mod 6, then the remainders are a permutation of the set $\{0, 1, 2, 3, 4, 5\}$ whose sum is

$$0 + 1 + 2 + 3 + 4 + 5 = 15 \equiv 3 \pmod 6.$$

Since $42 \not\equiv 3 \pmod 6$, we get a contradiction.

5. **There are several signs $+$ and $-$ on a blackboard. You may erase two signs and write, instead, $+$ if they are equal and $-$ if they are unequal. Prove that the last sign on the board does not depend on the order of erasure.**
The parity of the number of $-$ signs does not change:
   - if two $+$'s are replaced by $+$, then the number of $-$'s does not change,
   - if two $-$'s are replaced by $+$, then the number of $-$'s is decreased by 2,
   - if $+$ and $-$ are replaced by $-$, then the number of $-$'s does not change.

Therefore if we had an even number of $-$ signs then a $+$ will remain in the end, and if we had an odd number of $-$ signs then a $-$ will remain in the end.