Math 145 **Fall 2003**

Homework 6

Invariants

Due 13 October 2003, 5 points each:

1. Start with the positive integers $1, 2, \ldots, 4n-1$. In each step you may replace any two integers by their difference. Prove that an even integer will be left after $4n-2$ steps.
2. Start with the set $\{1, 2, 3, 4\}$. In each step you may add or subtract 2 times one of the numbers to/from another number. Say, you can replace 1 by $1+2\cdot 2$, or by $1-2\cdot 2$, or by $1+2\cdot 3$, etc. Can you reach $\{10, 20, 30, 40\}$?
3. Each of the numbers 1 to 10^6 is repeatedly replaced by its digital sum until we reach 10^6 one-digit numbers. For example, 987654 is replaced by $9+8+7+6+5+4 = 39$, then 39 is replaced by $3+9 = 12$, and finally, 12 is replaced by $1+2 = 3$. Among these 10^6 one-digit numbers, will we have more 1's or 2's?
4. The integers $1, 2, 3, 4, 5, 6$ are arranged in any order on 6 places numbered 1 through 6. Now we add its place number to each integer. Prove that there are two among the sums which have the same remainder mod 6.
5. There are several signs $+$ and $-$ on a blackboard. You may erase two signs and write, instead, $+$ if they are equal and $-$ if they are unequal. Prove that the last sign on the board does not depend on the order of erasure.

Extra credit: In the table below, you may switch the signs of all numbers of a row, column, or parallel to one of the diagonals, In particular, you may switch the sign of each corner square. Prove that at least one -1 will remain in the table.

-1	1	-1	1
1	1	1	1
1	1	-1	-1
1	-1	1	1