Homework 8 - Solutions

Case study

1. **Solve for x:** $x^{(x^2)} = x^2$.

   First check $x = 0$. This is not a root because 0^0 is undefined. If $x \neq 0$, we can divide both sides of the equation by x^2. We get $x^{x^2-2} = 1$. Consider 3 cases:

   Case I: $x = 1$, this is a root since (as is easy to check) it satisfies the original equation.

   Case II: $x \neq 0$, $x^2 - 2 = 0$ gives $x = \pm\sqrt{2}$. Both satisfy the original equation.

   Case III: $x = -1$, $x^2 - 2$ is even. This has no solutions because if $x = -1$ then $x^2 - 2 = -1$ which is not even.

   Answer: 1, $\sqrt{2}$, $-\sqrt{2}$.

2. **Find all the pairs (x, y) that satisfy the system** $\begin{cases} x^{2x} & = & y+1 \\ x^y & = & 1 \end{cases}$

   First consider the second equation. There are 3 cases:

   Case I: $x = 1$. Then the first equation gives $1^2 = y + 1$, so $y = 0$. Check again that $(1, 0)$ satistfies both equations.

   Case II: $x \neq 0$, $y = 0$. The first equation then becomes $x^{2x} = 1$. Consider 3 cases here:

   Case I: x=1 gives the same solution as the one we found above.

   Case II: $x \neq 0$, $2x = 0$ has no solutions.

   Case III: $x = -1$, $2x$ is ok, and then check again that $(-1, 0)$ satisfies both equations.

   Case III: $x = -1$, y even. The first equation then becomes $(-1)^{-2} = y + 1$, so $y = 0$. This gives $(-1, 0)$ again.

   Answer: $(1, 0)$ and $(-1, 0)$.

3. **Solve for x:** $x^2 - |5x - 6| \leq 0$.

   Case I: $5x - 6 \geq 0$, or equivalently $x \geq \frac{6}{5}$.

   Then the inequality becomes $x^2 - (5x - 6) \leq 0$.
   $x^2 - 5x + 6 \leq 0$
   $(x - 2)(x - 3) \leq 0$
   $x \in [2, 3]$

Since all the points in the interval $[2, 3]$ satisfy the condition $x \geq \frac{6}{5}$, all of them are solutions.

Case II: $5x - 6 < 0$, or $x < \frac{6}{5}$.

Then the inequality becomes $x^2 + (5x - 6) \leq 0$.
$x^2 + 5x - 6 \leq 0$
$(x + 6)(x - 1) \leq 0$
$x \in [-6, 1]$
Since all the points in the interval $[-6, 1]$ satisfy the condition $x < \frac{6}{5}$, all of them are solutions.

Answer: $[-6, 1] \cup [2, 3]$.

4. **Sketch the graph of** $f(x) = |x + |x + 2||$**.**

First sketch the graph of $y = x + |x + 2|$.

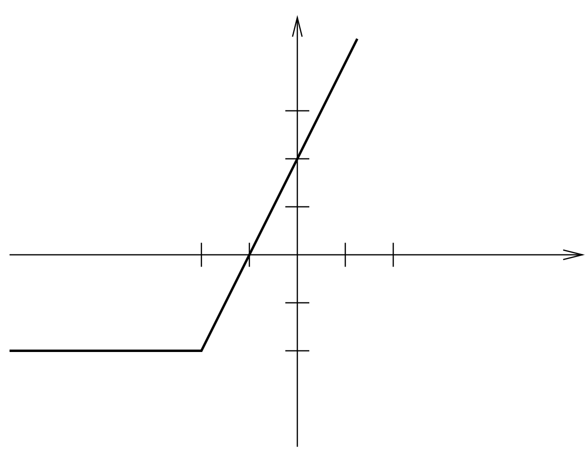
Case I: $x + 2 \geq 0$, or $x \geq -2$.

Then $y = x + x + 2 = 2x + 2$, so we draw the line $y = 2x + 2$ on the interval $[-2, \infty)$.

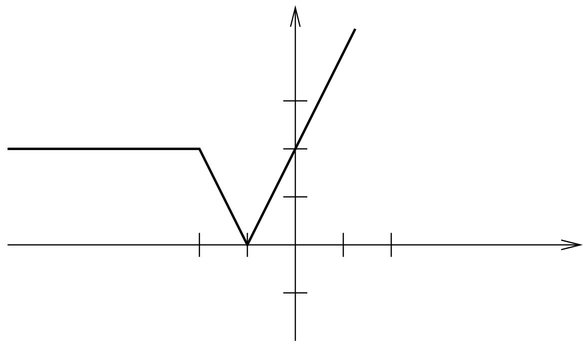
Case II: $x + 2 < 0$, or $x < -2$.

Then $y = x - (x + 2) = x - x - 2 = -2$, so we draw the horizontal line $y = -2$ on the interval $(-\infty, -2)$.

Thus we have the graph of $y = x + |x + 2|$:



Now we take the absolute value of the whole expression, and obtain the graph of $y = |x + |x + 2||$:



5. **Sketch the region** $\{(x, y) \mid |x| + |y^3| < 8\}$**.**

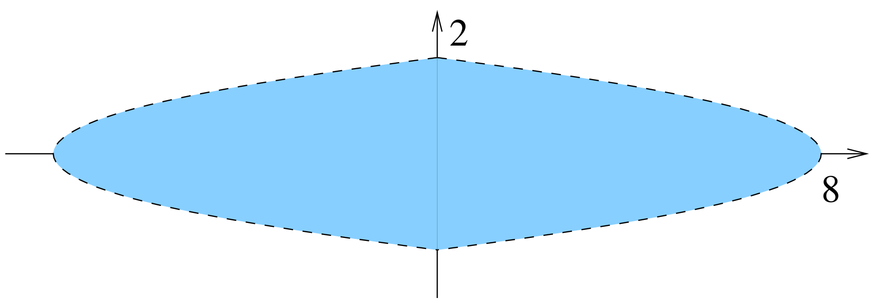
   Case I: $x \geq 0$, $y \geq 0$, then $x + y^3 < 8$, or $x < 8 - y^3$.

   Case II: $x \geq 0$, $y < 0$, then $x - y^3 < 8$, or $x < 8 + y^3$.

   Case III: $x < 0$, $y \geq 0$, then $-x + y^3 < 8$, or $x > y^3 - 8$.

   Case IV: $x < 0$, $y < 0$, then $-x - y^3 < 8$, or $x > -8 - y^3$.

   Now we draw the corresponding region in each quadrant, and we get the following figure:



   Note: since the inequality is strict, the boundary if the region is excluded.