Homework 9 - Solutions

Graphs

1. **Explain why a graph cannot have 7 vertices of degrees 4, 4, 3, 3, 3, 2, 2.**

   By the corollary on the first page of the handout, in any graph, the number of vertices of odd degree is even. Here there are 3 vertices of degree 3, so there is no such graph.

2. **Prove that in any group of people, the number of people that are friends with an odd number of people is even.**

   Let vertices represent people, and edges represent friendship (two vertices are connected if and only if the corresponding people are friends). Then the degree of each vertex is the number of friends of the corresponding person. Since in any graph, the number of vertices of odd degree is even, we have that the number of people with an odd number of friends is even.

3. **For what values of n and m does $K_{n,m}$ have**

   First of all, recall that $K_{n,m}$ has 2 groups of vertices, n vertices in group A, m vertices in group B, and every vertex in group A is connected to every vertex in group B.

   (a) **an Euler cycle?**

   We know that there exists an Euler cycle if and only if the degree of each vertex is even. The graph $K_{n,m}$ has n vertices of degree m and m vertices of degree n. Since all the degrees must be even, both m and n must be even.

   (b) **an Euler path?**

   By a problem discussed in class, an Euler path exists if and only if the graph has at most 2 vertices of odd degree. So consider the following cases:

   (1) No vertices of odd degree, i.e. all the degrees are even. Then both m and n are even.

   (2) 2 vertices of odd degree, both in group A of n vertices. Since all the vertices in this group have the same (odd) degree, and we can have at most 2 vertices of odd degree, there are only 2 vertices in this group, thus $n = 2$. Since their degree is odd, m is odd. Thus we have $n = 2$ and m is odd.

   (3) 2 vertices of odd degree, both in group B of m vertices. This case is similar to case (2), only n and m are switched. Thus $m = 2$ and n is odd.

   (4) 2 vertices of odd degree, one in group A and the other in group B. Then both m and n are odd, thus all the degrees are odd, but we can have at most 2 odd degrees, so $n = m = 1$.

   (c) **a Hamilton cycle?**

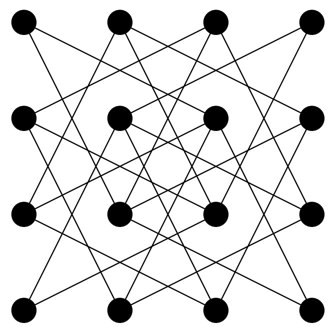
   A Hamilton cycle is a cycle that visits every vertex exactly once. If a Hamilton cycle starts at a vertex in group A, then its second vertex belongs to group B, the next one belongs to group A, the fourth one belongs to group B, and so on, i.e. A and B will alternate. It must eventually come back to the original vertex, therefore the number of vertices in group A must be equal to the number of vertices in group B. Thus $m = n$.

(d) **a Hamilton path?**

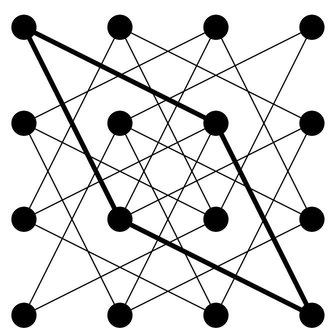
A path does not return to the starting point, thus in addition to the case $m = n$ (in this case a path has the form ABAB...AB), we have $m = n - 1$ (then we can find a path of the form ABAB...ABA), and $m = n + 1$ (then we can find a path of the form (BABA...BAB).

4. **Show that there is no reentrant knight's tour on a 4×4 chessboard. Actually, there is no tour at all, but that is a bit harder to prove.**

First draw the graph representing all possible moves of a knight:



A reentrant tour is a Hamilton cycle. Thus we have to show that this graph has no Hamilton cycle. Notice that there are 4 vertices of degree 2, and in order to visit a vertex of degree 2 we have to use both its edges. Consider the upper left corner vertex and the lower right corner vertex. We must use both edges at each of them. But then we get a cycle. There is no way of adding anything to this cycle (because if we add more edges, we'll have to go through some vertex more than once). But this cycle misses many vertices. Thus there is no Hamilton cycle.



5. **There are 7 men and 7 women attending a dance. After the dance, they recall the number of people they have danced with. The numbers are as follows: 3, 3, 3, 3, 3, 3, 3, 5, 6, 6, 6, 6, 6, 6. Prove that at least one of them made a mistake. (Assume that men only danced with women, and women only danced with men.)**

If nobody made a mistake, we would be able to draw a bipartite graph with 14 vertices, 7 vertices representing men and 7 vertices representing women, and such that 2 vertices are connected if and only if the corresponding people shared a dance. Then the sum of the degrees of the 7 vertices representing men should be equal to the sum of the degrees of the 7 vertices representing women (both sums being equal to the number of edges). But it is not possible to divide the given 14 numbers into 2 groups such that the sums are equal because one group must contain the 5, and the other group must consist of 3's and 6's. The sum of the numbers in the first group is congruent to 2 mod 3, and the sum of the numbers in the second group is congruent to 0 mod 3, so the sums cannot be equal.