Mathematical Induction

Problems

Prove the following statements.

1. For any positive integer n, $n < 2^n$.

2. If $m = 2^q$ where q is a positive integer, then $3^m - 1$ is divisible by 2^{q+2}.

3. Suppose that $2n$ points are given in space. Altogether $n^2 + 1$ line segments are drawn between these points. Then there is at least one set of three points which are joined pairwise by line segments.

4. Let $\{F_0,\ F_1,\ F_2,\ \ldots\}$ be the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$, $n \geq 1$. Then

   (a) $\displaystyle\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$

   (b) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$

5. There are n identical cars on a circular track. Among all of them, they have just enough gas for one car to complete a lap. Show that there is a car which can complete a lap by collecting gas from other cars on its way around.

6. Let α be any real number such that $\alpha + \dfrac{1}{\alpha} \in \mathbb{Z}$. Then $\alpha^n + \dfrac{1}{\alpha^n} \in \mathbb{Z}$ for any $n \in \mathbb{N}$.

7. Suppose that n lines are given in the plane. They divide the plane into parts. Show that you can color the plane with two colors, so that no parts with a common boundary line are colored the same way. Such a coloring is called a proper coloring.

8. Consider all possible subsets of the set $\{1, 2, \ldots, N\}$, which do not contain any neighboring elements. Then the sum of the squares of the products of all numbers in these subsets is $(N+1)!-1$. (e.g. if $N = 3$, then $1^2+2^2+3^2+(1\cdot 3)^2 = 23 = 4!-1$.)

9. Find the determinant of the $n \times n$ matrix A_n with entries $a_{ij} = \begin{cases} 2 \text{ if } i = j \\ 1 \text{ if } |i-j| = 1 \\ 0 \text{ otherwise} \end{cases}$

   Prove your formula using Mathematical Induction.