Invariants

Examples

Find an invariant, i.e. something that doesn’t change:

**Problem.** The numbers 1, 2, \ldots, 10 are written on the blackboard. Then we pick any two numbers, \(a\) and \(b\), erase them, and write \(a + 1\) and \(b - 1\) instead. Is it possible to get ten 5’s this way?

**Solution.** Notice that when we increase \(a\) by 1 and decrease \(b\) by 1, the sum of the numbers does not change. Initially the sum is \(1 + 2 + \ldots + 10 = 55\), and \(10 \cdot 5 = 50\), so it is not possible to get ten 5’s.

**Problem.** Each of the numbers \(a_1, a_2, \ldots, a_n\) is 1 or \(-1\), and

\[
S = a_1a_2a_3a_4 + a_2a_3a_4a_5 + a_2a_3a_4 + \ldots + a_{n-1}a_n a_1 a_2 + a_n a_1 a_2 a_3 = 0.
\]

Prove that \(4|n\).

**Solution.** If we replace \(a_i\) by \(-a_i\), then \(S\) does not change mod 4 since four terms change their sign. Indeed, if all four terms are of the same sign, then their sum changes from 4 to \(-4\) or from \(-4\) to 4, thus \(S\) changes by \(\pm 8\). If one or three have the same sign, then their sum changes from 2 to \(-2\) or from \(-2\) to 2, thus \(S\) changes by \(\pm 4\). Finally, if two are positive and two are negative, then the sum doesn’t change. Initially, we have \(S = 0\) which implies \(S \equiv 0\) (mod 4). Now, step-by-step, we change each negative \(a_i\) into a positive 1. This does not change \(S\) mod 4. At the end, we still have \(S \equiv 0\) (mod 4), but also \(S = n\), i.e., \(4|n\).