Invariants

Examples

Find an invariant, i.e. something that doesn't change:

Problem. The numbers $1, 2, \ldots, 10$ are written on the blackboard. Then we pick any two numbers, a and b, erase them, and write $a+1$ and $b-1$ instead. Is it possible to get ten 5's this way?

Solution. Notice that when we increase a by 1 and decrease b by 1, the sum of the numbers does not change. Initially the sum is $1 + 2 + \ldots + 10 = 55$, and $10 \cdot 5 = 50$, so it is not possible to get ten 5's.

Problem. Each of the numbers $a_1, a_2, \ldots, a_n$ is 1 or -1, and

$$S = a_1a_2a_3a_4 + a_2a_3a_4a_4 + a_2a_3a_4a_5 + \ldots a_{n-1}a_na_1a_2 + a_na_1a_2a_3 = 0.$$

Prove that $4|n$.

Solution. If we replace a_i by $-a_i$, then S does not change mod 4 since four terms change their sign. Indeed, if all four terms are of the same sign, then their sum changes from 4 to -4 or from -4 to 4, thus S changes by ± 8. If one or three have the same sign, then their sum changes from 2 to -2 or from -2 to 2, thus S changes by ± 4. Finally, if two are positive and two are negative, then the sum doesn't change. Initially, we have $S = 0$ which implies $S \equiv 0 (\text{mod } 4)$. Now, step-by-step, we change each negative a_i into a positive 1. This does not change S mod 4. At the end, we still have $S \equiv 0 (\text{mod } 4)$, but also $S = n$, i.e., $4|n$.