Math 145 **Fall 2003**

Invariants

Problems

1. Let n be an odd positive integer. First we write the numbers $1, 2, 3, \ldots, 2n$ on the blackboard. Then we pick any two numbers, a and b, erase them, and write, instead, $|a - b|$. We do this until only one number remains. Prove that an odd number will remain at the end.
2. Initially 1 is written in every cell of a 5×5 table. You may change the sign of the numbers in any two adjacent cells. Is it possible to make all of the numbers -1?
3. Start with the set $\{1, 2, 3, 4, 5, 6\}$. In each step, you may add 2 to any 5 numbers or subtract 1 from any 5 numbers. Can you reach $\{1, 2, 4, 8, 16, 32\}$?
4. Assume we have an 8×8 chessboard with the usual coloring. You may repaint all squares of a row or column. The goal is to attain just one black square. Can you reach the goal? What if you are allowed to repaint all squares of a 2×2 square?
5. Start with the set $\{1, 3, 6\}$. In each step you may choose two of the numbers, let's call them a and b, and replace them by $0.6a - 0.8b$ and $0.8a + 0.6b$. Can you reach $\{2, 4, 5\}$?
6. You write all the digits from 1 to 9 in a row in any order you like, and then write plus signs between some digits (as many plus signs as you like). For example, you could write $7 + 35 + 19 + 7 + 4 + 2 + 8 + 6$. Finally, you evaluate of the obtained expression. Prove that there is no way to get the value of 100. Or 101. Or 102. Or 103... What is the smallest possible three-digit number that can be obtained in this game?
7. A circle is divided into six sectors. Then the numbers 1, 0, 1, 0, 0, 0 are written into the sectors. You may increase any two neighboring numbers by 1. Is it possible to equalize all numbers by a sequence of such steps?
8. There are a white, b black, and c red chips on a table. In one step, you may choose two chips of different colors and replace them by one chip of the third color. If just one chip will remain at the end, prove that its color does not depend on the evolution of the game, but it only depends on the numbers a, b, and c.