Logic and types of proofs

Problems

1. Show that the following propositions are logically equivalent.
   (a) $p \to q$ and $\neg p \vee q$.
   (b) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$.

2. Translate the statement

$$\forall x(C(x) \vee \exists y(C(y) \wedge F(x,y)))$$

   into English, where $C(x)$ is "x has a computer", $F(x,y)$ is "x and y are friends", and the domain for both x and y is the set of all students in Fresno.

3. Let $Q(x,y)$ denote "$x + y = 0$". What are the truth values of the statements $\exists y \forall x Q(x,y)$ and $\forall x \exists y Q(x,y)$?

4. Express the definition of the limit $\lim_{x \to a} f(x) = L$ using quantifiers.

5. Express the definition of a convergent sequence $a_1, a_2, \ldots$ using quantifiers.

6. Let $F(x,y)$ be statement "x can fool y", where the domain for both variables is the set of all people in the world. Use quantifiers to express each of the following statements:
   (a) Everybody can fool Fred.
   (b) Mike can fool everybody.
   (c) Everybody can fool somebody.
   (d) There is no one who can fool everybody.
   (e) Everyone can be fooled by somebody.
   (f) No one can fool both Fred and Jerry.
   (g) Nancy can fool exactly two people.
   (h) There is exactly one person whom everybody can fool.
   (i) No one can fool himself or herself.
   (j) There is someone who can fool exactly one person besides himself or herself.

7. Rewrite each of the following statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

   (a) $\neg\forall x \forall y P(x, y)$
   (b) $\neg\forall y \exists x P(x, y)$
   (c) $\neg\forall y \forall x (P(x, y) \vee Q(x, y))$
   (d) $\neg(\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$
   (e) $\neg\forall x (\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z))$

8. Prove that for any number n there is a prime number greater than n. Is your proof constructive? Does it use Mathematical Induction?

9. Prove or disprove that $2^n + 1$ is prime for all nonnegative integers n.

10. Prove that if the sum of two numbers is irrational then at least one of the numbers is irrational. Is your proof direct, by contradiction, or by contrapositive? State the converse. Prove or disprove the converse.

11. Prove or disprove that if a and b are rational numbers, then a^b is also rational.

12. Prove that the equation $4 \sin^2 x = 1$ has a real solution. Is your proof constructive?

13. Prove that the equation $x + \sin x = 1$ has a real solution. Is your proof constructive?

14. Prove that the equation $x^2 + x + 1 = 0$ has no rational solutions. Is your proof direct, by contradiction, or by contrapositive?

15. Prove that an integer number a is even if and only if a^2 is even. Did you prove the two implications separately or simultaneously?

16. Prove that 0 is a root of the equation $a_n x^n + \ldots a_1 x + a_0 = 0$ if and only if the free term $a_0 = 0$. Did you prove the two implications separately or simultaneously?