Math 145 **Fall 2003**

Practice Test 1 - Solutions

Answer the question (5 points):

- What does "a and b are relatively prime" mean?
Answer: *It means that the greatest common divisor of a and b is 1.*
or: *It means that no number bigger than 1 divides both a and b.*

and do any 3 of the following problems (15 points each):

1. Prove that if n is an integer then $n^2 + 2$ is not divisible by 5.
Solution: *Consider all possible remainders of n mod 5, and in each case, compute the remainder of $n^2 + 2$:*
If $n \equiv 0 \pmod 5$, then $n^2 + 2 \equiv 2 \pmod 5$;
if $n \equiv 1 \pmod 5$, then $n^2 + 2 \equiv 3 \pmod 5$;
if $n \equiv 2 \pmod 5$, then $n^2 + 2 \equiv 6 \equiv 1 \pmod 5$;
if $n \equiv 3 \pmod 5$, then $n^2 + 2 \equiv 11 \equiv 1 \pmod 5$;
if $n \equiv 4 \pmod 5$, then $n^2 + 2 \equiv 18 \equiv 3 \pmod 5$;
We see that $n^2 + 2$ is never congruent to 0 mod 5, so it is never divisible by 5.
Another solution: *We know from a homework problem that n^2 can end with 0, 1, 4, 5, 6, or 9. Then $n^2 + 2$ can end with 2, 3, 6, 7, 8, or 1. So it never ends with 5 or 0, therefore it is never divisible by 5.*

2. Prove that for any natural n,
$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1.$$
Solution: *We will prove this identity by Mathematical Induction.*
Basis step: for $n = 1$ we have $1 \cdot 1! = 2! - 1$, or $1 = 2 - 1$ which is true.
Inductive step: suppose the identity holds for $n = k$, i.e.
$$1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1.$$
Add $(k+1) \cdot (k+1)!$ to both sides:
$1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k+1) \cdot (k+1)! = (k+1)! - 1 + (k+1) \cdot (k+1)! =$
$= (k+1)!(1 + k + 1) - 1 = (k+2)! - 1.$
Thus the identity holds for $n = k + 1$.

3. Let $P(x, y)$ denote the proposition "$x < y$" where x and y are real numbers. Determine the truth values of
   (a) $\exists x \exists y P(x, y)$,
   (b) $\forall x \exists y P(x, y)$,
   (c) $\exists x \forall y P(x, y)$,
   (d) $\forall x \forall y P(x, y)$.

Solution:

(a) True. Example: $x = 1$, $y = 2$, $1 < 2$.

(b) True. For any x, *if we take* $y = x + 1$ *then* $x < y$.

(c) False. There is no such x *that for any* y, $y > x$, *because for any* x *we can take* $y = x$, *then* $x \not< y$.

(d) False. Counterexample: $x = 2$, $y = 1$, $2 \not< 1$.

4. Let $f_1(x) = 2x + 1$ and $f_n = f_1 \circ f_{n-1}$. Compute f_n for some small values of n. Notice the pattern. Write a formula for f_n and prove it using Mathematical Induction.

Solution:

$f_2(x) = 2(2x + 1) + 1 = 4x + 3$

$f_3(x) = 2(4x + 3) + 1 = 8x + 7$

$f_4(x) = 2(8x + 7) + 1 = 16x + 15$

It appears that $f_n = 2^n x + 2^n - 1$, *so we will try to prove this by Mathematical Induction.*

Basis step: if $n = 1$, *then our formula gives* $f_1 = 2x + 1$ *which is true.*

Inductive step: suppose the formula holds for $n = k$, *i.e.* $f_k = 2^k x + 2^k - 1$. *Then* $f_{k+1} = f_1 \circ f_k = 2(2^k x + 2^k - 1) + 1 = 2^{k+1} x + 2^{k+1} - 2 + 1 = 2^{k+1} x + 2^{k+1} - 1$. *Thus the formula holds for* $n = k + 1$.

Extra credit (15 points):

- Six points are selected inside a 3×4 rectangle. Prove that there two of them such that the distance between them is at most $\sqrt{5}$.

  Solution: *Divide the rectangle into 5 regions as shown on the picture. Since there are 6 points, by Dirichlet's principle at least two of them are in the same region. The distance between them is at most* $\sqrt{5}$.