Practice Test 2 - Solutions

Answer the question (5 points):

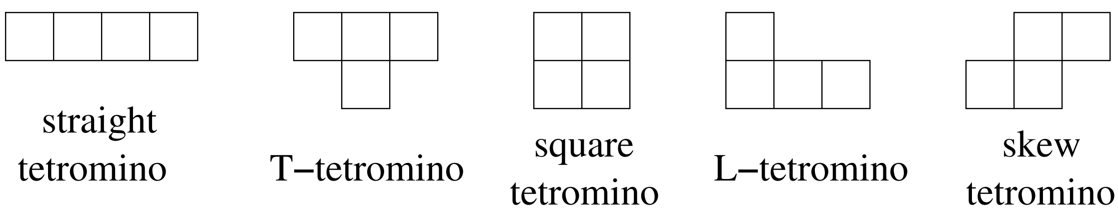
- What is an Euler cycle?

Answer: *An Euler cycle is a cycle containing every edge exactly once.*

and do any 3 of the following problems (15 points each):

1. Start with the set $\{1, 1, 1, 1\}$. In each step, you may either multiply one of the numbers by 3, or subtract 2 from it. Show that it is not possible to reach the set $\{1, 2, 3, 4\}$.
   Solution: *An odd number times 3 is an odd number, and an even number times 3 is an even number. So multiplication by 3 does not change the parity of the number. Also, an odd number minus 2 is an odd number, and an even number minus 2 is an even number. So neither of these operations changes the parity of the number. The initial set consists of four odd numbers. Thus the four numbers will always be odd. It is not possible to reach 2 odd and 2 even numbers.*

2. An 8×8 chessboard is covered by tetrominoes (all 5 kinds are shown below). Prove that the number of T-tetrominoes is even.



   Solution: *Consider the traditional coloring of the chessboard. A T-tetromino covers either 1 or 3 (i.e. an odd number of) black squares. Every other tetromino covers 2 (i.e. an even number of) black squares. If the number of T-tetrominoes were odd, then they would cover an odd number of black squares (because the sum of an odd number of odd numbers is odd). But the chessboard has 32 black squares, and 32 is even. Contradiction.*

3. Solve for x: $|x-5| + |2x-4| \leq 6$.
   Solution: <u>Case I:</u> $x - 5 \geq 0$ and $2x - 4 \geq 0$, so $x \geq 5$ and $x \geq 2$, which is equivalent to $x \geq 5$.
   Then we have $(x-5) + (2x-4) \leq 6$
   $3x - 9 \leq 6$
   $3x \leq 15$
   $x \leq 5$.
   Together with the condition $x \geq 5$ this gives $x = 5$.
   <u>Case II:</u> $x - 5 \geq 0$ and $2x - 4 < 0$, so $x \geq 5$ and $x < 2$, which is impossible.
   <u>Case III:</u> $x - 5 < 0$ and $2x - 4 \geq 0$, so $x < 5$ and $x \geq 2$, which is equivalent to $2 \leq x < 5$.
   In this case we have $-(x-5) + (2x-4) \leq 6$
   $-x + 5 + 2x - 4 \leq 6$
   $x + 1 \leq 6$
   $x \leq 5$.
   Together with the condition $2 \leq x < 5$ this gives the interval $[2, 5)$.

Case IV: $x - 5 < 0$ and $2x - 4 < 0$, so $x < 5$ and $x < 2$, which is equivalent to $x < 2$.
Now we have $-(x-5) - (2x-4) \leq 6$
$-x + 5 - 2x + 4 \leq 6$
$-3x + 9 \leq 6$
$-3x \leq -3$
$x \geq 1$
Together with the condition $x < 2$ this gives the interval $[1, 2)$.
Thus the solution consists of the point 5, the interval $[2, 5)$, and the interval $[1, 2)$. The union of these is the interval $[1, 5]$.

4. There are 8 counties in Sikinia. There are no "four corners" points (like Arizona, Colorado, New Mexico, and Utah). Each county counted the number of neighboring counties. The numbers are 5, 5, 4, 4, 4, 4, 4, 3. Prove that at least one county made a mistake.
Solution: *If this were possible, consider the following graph with 8 vertices: each vertex represents a county, and two vertices are connected if and only if the corresponding counties are neighbors. Then the degree of each vertex is the number of the neighbors of that county. Thus we would have a graph with 8 vertices of degrees 5, 5, 4, 4, 4, 4, 4, 3. But in any graph, the sum of the degrees of all the vertices is even. The sum* $5+5+4+4+4+4+4+3 = 33$ *is odd. Contradiction.*

Extra credit (15 points):

- Nine 1×1 cells of a 10×10 square are infected. In one time unit, the cells with at least two infected neighbors (having a common side) become infected. Can the infection spread to the whole square?
Solution: *Consider squares with 2, 3, or 4 infected neigbors. When the infection spreads to such a square, notice that the perimeter of the contaminated area cannot increase (but it may decrease). Namely (look at the picture below), when a square with 2 infected neigbors becomes infected, the perimeter of the contaminated area does not change. When a square with 3 infected neigbors becomes infected, the perimeter decreases by 2. When a square with 4 infected neigbors becomes infected, the perimeter decreases by 4. Initially the perimeter is at most* $4 \cdot 9 = 36$. *It cannot become 40.*