**Math 145** **Fall 2003**

# Practice Test 3 - Solutions

**Evaluate the integral** (5 points):

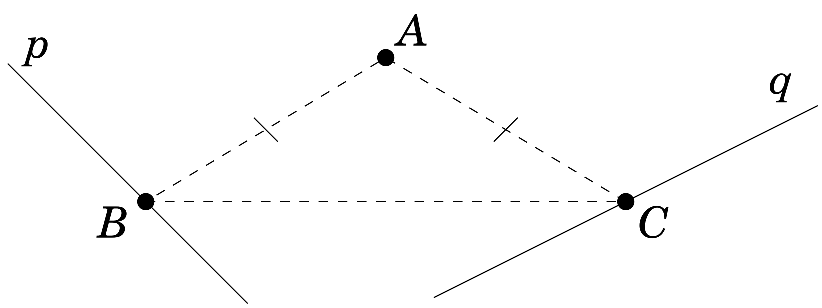
- $\int_{-1}^{1} |x|\,dx$

**Solution 1:** $\int_{-1}^{1} |x|\,dx = \int_{-1}^{0} |x|\,dx + \int_{0}^{1} |x|\,dx = -\int_{-1}^{0} xdx + \int_{0}^{1} xdx =$
$-\left.\frac{x^2}{2}\right|_{-1}^{0} + \left.\frac{x^2}{2}\right|_{0}^{1} = -\left(-\frac{1}{2}\right) + \frac{1}{2} = 1.$

**Solution 2:** *The value of the integral is the area of the region under the graph of $f(x) = |x|$ from $-1$ to $1$. This region consists of two triangles with base 1 and height 1, thus the area of each triangle is $\frac{1}{2}$. The total area is then equal to 1.*

**and do any 3 of the following problems** (15 points each):

1. Given a point $A$, and two lines $p$ and $q$, find a point $B$ on $p$ and a point $C$ on $q$ such that the triangle $ABC$ is isosceles with $AB = AC$, and the base $BC$ is horizontal. Assume that a solution exists.




**Solution:** *Draw a vertical line $l$ through $A$. Reflect the line $p$ about $l$, get $p'$. Let $C$ be the intersection point of $p'$ and $q$. Now reflect the point $C$ about $l$, get a point $B$ on $p$. Since $B$ and $C$ are symmetric about the vertical line $l$, we have that $BC$ is horizontal and $AB = AC$.*

2. Find the greatest common divisor $d$ of $a = 46$ and $b = 32$, and integer numbers $x$ and $y$ such that $xa + yb = d$.
**Solution:**
$46 = 1 \cdot 32 + 14$
$32 = 2 \cdot 14 + 4$
$14 = 3 \cdot 4 + 2$
$4 = 2 \cdot 2$
*Thus* $d = (46, 32) = 2$.

$$
\begin{aligned}
2 &= 14 - 3 \cdot 4 \\
&= 14 - 3(32 - 2 \cdot 14) = 7 \cdot 14 - 3 \cdot 32 \\
&= 7(46 - 1 \cdot 32) - 3 \cdot 32 = 7 \cdot 46 - 10 \cdot 32
\end{aligned}
$$

*Thus* $x = 7$ *and* $y = -10$.

3. Starting with 2, 0, 0, 3, we construct the sequence 2, 0, 0, 3, 5, 8, 6, ..., where each new digit is the mod 10 sum of the preceding four terms. Prove that the 4-tuple 0, 5, 0, 5 will never occur.
   **Solution:** *Suppose the 4-tuple 0, 5, 0, 5 occurs. Then before it we must have the digit 0, and before that 5, and before that another 0... In fact, all the digits in our sequence must be 0's and 5's.*
   *Proof: Solving* $a_n \equiv a_{n-4} + a_{n-3} + a_{n-2} + a_{n-1} \pmod{10}$ *for* $a_{n-4}$ *gives*
   $a_{n-4} \equiv a_n - a_{n-3} - a_{n-2} - a_{n-1} \pmod{10}$.
   *This implies* $a_{n-4} \equiv a_n - a_{n-3} - a_{n-2} - a_{n-1} \pmod{5}$.
   *Thus if four consecutive digits are divisible by 5, then all the digits in the sequence are divisible by 5.*
   *But the starting sequence 2, 0, 0, 3 contains 2 and 3 which are not divisible by 5. Contradiction.*

4. Find a number $c$ such that the line $y = x - 1$ is tangent to the parabola $y = cx^2$.
   **Solution:** *Let the given line be tangent to the parabola at the point* $(a, a-1)$. *Then first, the parabola passes through* $(a, a-1)$, *thus*

   $$a - 1 = ca^2.$$

   *Second, the line and the parabola have the same slope at this point:*

   $$1 = 2ca.$$

   *From the second equation we have* $c = \dfrac{1}{2a}$. *Substitute this for c in the first equation:*

   $$a - 1 = \frac{a^2}{2a} \quad \Rightarrow \quad a - 1 = \frac{a}{2} \quad \Rightarrow \quad 2a - 2 = a \quad \Rightarrow \quad a = 2 \quad \Rightarrow \quad c = \frac{1}{4}$$

**Extra credit** (15 points):

- A sequence $\{a_n\}$ is defined recursively by the equations

  $$a_0 = a_1 = 1 \qquad n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}.$$

  Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} a_n$.

  **Sketch:** *Calculate first few terms, and notice that* $a_n = \dfrac{1}{n!}$. *Prove this formula by induction. Then* $\displaystyle\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{n!} = e$.