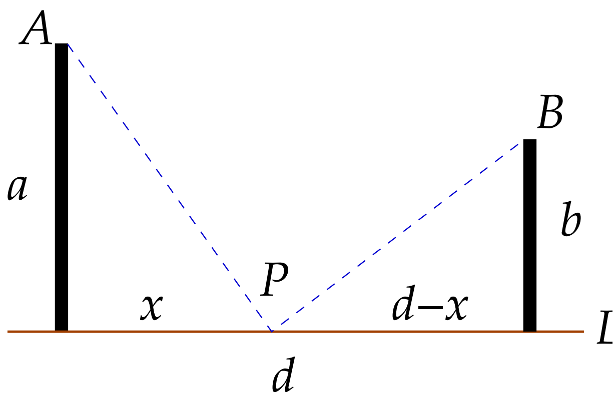
Symmetry, Translations, Rotations, and Similarity

Examples

Some geometry problems can be solved algebraically. Consider the following example.

Problem. There are 2 poles of heights a and b as shown below. The distance between the poles is d. Find the point on the ground equidistant from the tops of the poles.

Note. Depending on our goal, the word "find" here could mean either "calculate the location of that point", e.g. "find the distance between the point and one of the poles", or it could mean "find this point geometrically, using a ruler and a compass".



Calculation. Let x be the distance between one of the poles and the point P we are looking for. Then the distance between the other pole and the point P is $d-x$. We use Pythagorean theorem to compute the distances between P and the tops of the poles A and B, and set the two distances equal:

$$\sqrt{a^2+x^2} = \sqrt{b^2+(d-x)^2}$$

$$a^2+x^2 = b^2+(d-x)^2$$

$$a^2+x^2 = b^2+d^2-2dx+x^2$$

$$a^2 = b^2+d^2-2dx$$

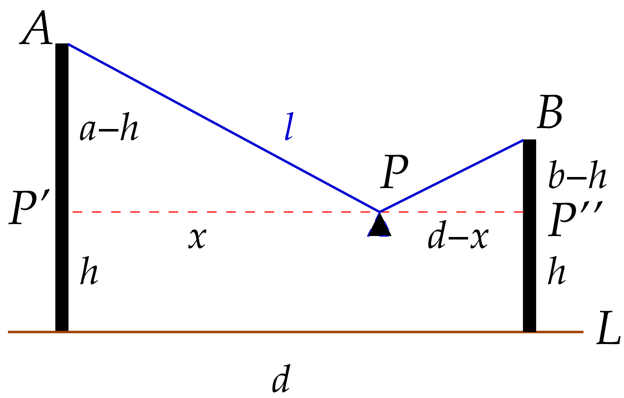
$$2dx = b^2+d^2-a^2$$

$$x = \frac{b^2+d^2-a^2}{2d}$$

Construction. Since the point P is equidistant from the tops of the poles, it lies on the perpendicular bisector of AB. Thus all we have to do is to draw the perpendicular bisector of AB, and then P is its intersection with the line L.

Remark. It is easy to calculate the position of the point P using the above construction. Introduce a coordinate system with the origin at the bottom of one of the poles, write an equation of the line through A and B, then write an equation of the perpendicular bisector, and find its x-intercept. Some problems are easier to solve by a geometric construction and a calculation based on the construction than by equations.

Problem. A rope of length l is strung between the two pole tops and a weight is hung from a ring on the rope, which is not long enough for the weight to reach the ground. How high from the ground does the weight hang?



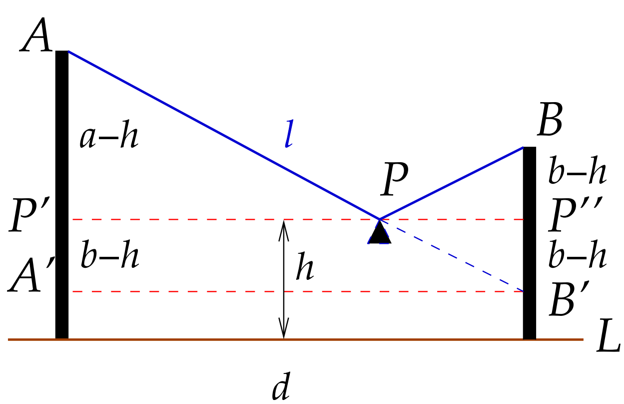
Solution.

Using Pythagorean theorem, we get $\sqrt{(a-h)^2+x^2}+\sqrt{(b-h)^2+(d-x)^2}=l$.

Using similar triangles APP' and BPP'', we get $\dfrac{a-h}{x}=\dfrac{b-h}{d-x}$.

Thus we have a system of two equations with two unknowns. Although it is possible to solve this system, it is not easy. There is a nicer way to solve this problem.

First find the location of the weight geometrically:



Now look at the triangle $A'B'A$. Since $A'B'=d$, $A'A=a+b-2h$, and $AB'=l$, we have

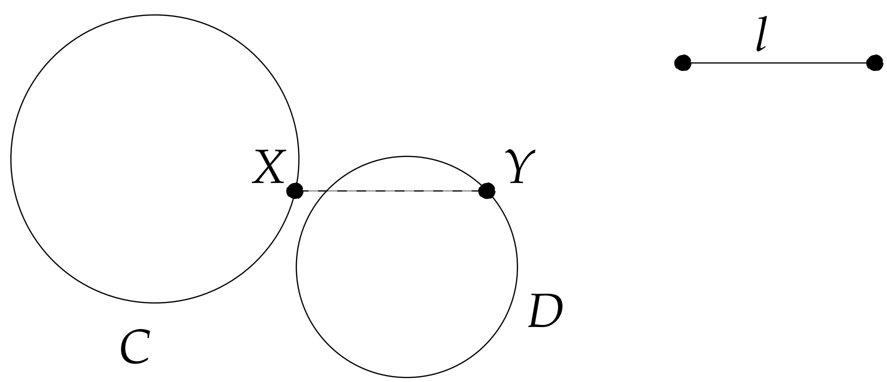
$d^2+(a+b-2h)^2=l^2$

$a+b-2h=\sqrt{l^2-d^2}$

$2h=a+b-\sqrt{l^2-d^2}$

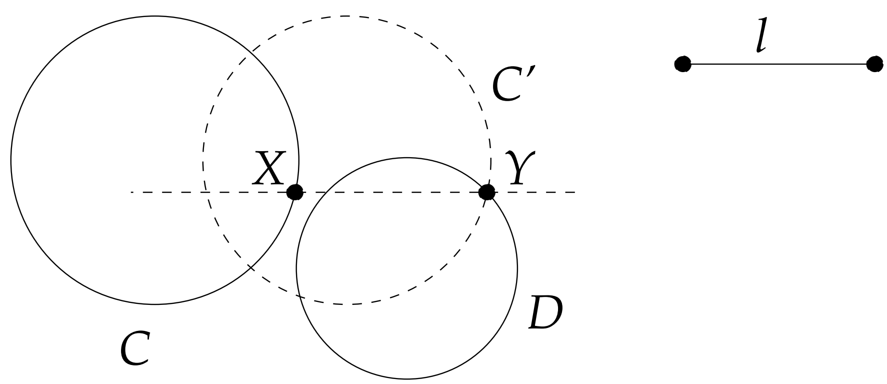
$h=\dfrac{a+b-\sqrt{l^2-d^2}}{2}$

Problem. Two circles C and D, and a distance l ale given. Draw a horizontal segment XY of length l such that X lies on C and Y lies on D. (Assume that such a segment exists.)



Solution.

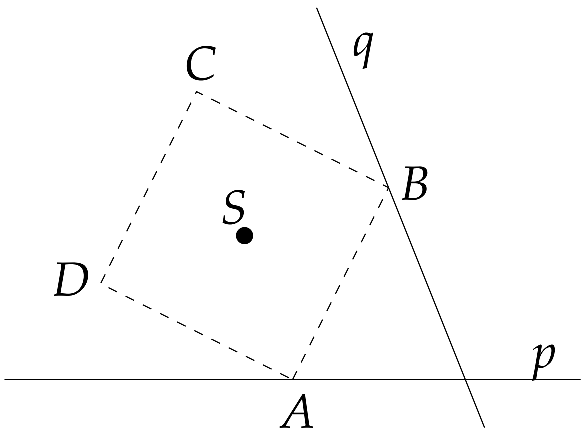
Translate the circle C by the distance l to the right. Let's call this new circle C'. Let Y be an intersection point of C' and D if it exists. Draw a horizontal line through Y. Then X is one of the intersection points of this line and the original circle C:



Note. If C' and D do not intersect, translate C' to the left instead of to the right.

Problem. Two distinct lines p and q are given, and a point S. Draw a square $ABCD$ that satisfies the following conditions:

- Point S is the center of the square.
- The vertex A of the square lies on the line p.
- The vertex B, the counterclockwise neighbor-vertex of A, lies on the line q.



Solution. Notice that the segments SA and SB must be perpendicular and of the same length. So rotate the line p through an angle of 90 degrees in the counterclockwise direction around the point S. Let p' be the new line. Let B be the intersection point of p' and q. Once we have one vertex and the center, it's easy: draw the line BS, find D such that $SD = SB$. Draw the line through S perpendicular to SB, find A and C.