MATH 145

Test 1 - Solutions

- Let $P(x, y)$ be a propositional function. Are $\forall x \exists y P(x, y)$ and $\exists y \forall x P(x, y)$ logically equivalent?
  Answer ("yes" or "no"): *No*

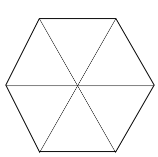
1. Prove that among 120 integers, there are two whose difference ends with 00.
   There are 100 possible remainders mod 100: 0, 1, 2, ..., 99. Since we have 120 (more than 100) numbers, by Dirichlet's box principle there are at least two numbers with the same remainder. Their difference has remainder 0, and thus is divisible by 100. Therefore it ends with two zeros.

2. Compute $A_n = 1 + 3 + 5 + \ldots + (2n - 1)$ for some small values of n. Notice the pattern. Write a formula for A_n and prove it using Mathematical Induction.
   $A_1 = 1$
   $A_2 = 1 + 3 = 4$
   $A_3 = 1 + 3 + 5 = 9$
   $A_4 = 1 + 3 + 5 + 7 = 16$
   We guess that $A_n = n^2$.
   Proof by Mathematical Induction:
   Basis step: for $n = 1$ *we have* $A_1 = 1^2 = 1$ *which is true.*
   Inductive step: suppose that the formula hold for $n = k$, *i.e.* $A_k = 1 + 3 + \ldots + (2k - 1) = k^2$.
   Then $A_{k+1} = 1 + 3 + \ldots + (2k - 1) + (2k + 1) = A_k + (2k + 1) = k^2 + 2k + 1 = (k + 1)^2$, *thus the formula holds for* $n = k + 1$.

3. Prove that for every integer n, $n^3 + 2n$ is divisible by 3.
   Proof 1:
   Consider the following 3 cases:
   - *if* $n \equiv 0 \pmod 3$, *then* $n^3 + 2n \equiv 0^3 + 2 \cdot 0 \equiv 0 \pmod 3$.
   - *if* $n \equiv 1 \pmod 3$, *then* $n^3 + 2n \equiv 1^3 + 2 \cdot 1 \equiv 3 \equiv 0 \pmod 3$.
   - *if* $n \equiv 2 \pmod 3$, *then* $n^3 + 2n \equiv 2^3 + 2 \cdot 2 \equiv 12 \equiv 0 \pmod 3$.

   We see that in each case $n^3 + 2n \equiv 0 \pmod 3$, *thus it is divisible by 3.*
   Proof 2:
   $n^3 + 2n \equiv n^3 + 2n - 3n \equiv n^3 - n \equiv n(n^2 - 1) \equiv (n - 1)n(n + 1) \pmod 3$.
   $(n - 1)n(n + 1)$ *is the product of 3 consecutive integers. One of them must be divisible by 3, thus the product is divisible by 3.*

4. 7 points are selected inside a regular hexagon whose sides have length 1. Prove that there are two points such that the distance between them is at most 1.



Divide the hexagon into 6 regions as shown in the figure. Since we have 7 (more than 6) points, by Dirichlet's box principle there is a region with at least two points in it (or on its boundary). The distance between those two points is at most 1 because each region is an equilateral triangle with all sides of length 1.

- **Extra credit:** Prove that among $n + 1$ positive integers all less than or equal to $2n$, there are two which are relatively prime.
  Divide the numbers $\{1, 2, \ldots, 2n\}$ *into* n *pairs of consecutive integers:* $\{1, 2\}, \{3, 4\}, \ldots \{2n - 1, 2n\}$. *Since we have* $n + 1$ *integers, at least two of them are consecutive. Their greatest common divisor is 1 because if a number* p *divides both* k *and* $k + 1$ *then* p *divides their difference* $(k + 1) - k = 1$, *so* $p = 1$. *Thus our two consecutive numbers are relatively prime.*