MATH 145

Test 2 - Solutions

- What is a Hamilton path?

  A Hamilton path is a path that goes through every vertex exactly once.

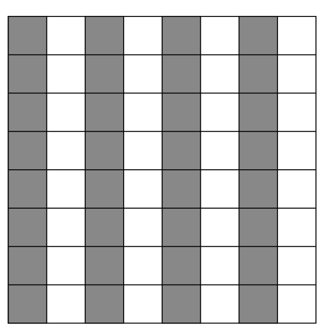
1. Start with the set $\{-3, -2, -1, 1, 2, 3\}$. In each step you may choose any two of these numbers and change their signs. Show that it is not possible to reach the set $\{3, 2, 1, 1, 2, 3\}$.

   Proof 1: When we change the sign of 2 numbers, the possibilities are:
   pos, pos $\rightarrow$ neg, neg
   pos, neg $\rightarrow$ neg, pos
   neg, neg $\rightarrow$ pos, pos.
   We see that the number of positive numbers either does not change or changes by 2. Thus the parity of the number of positive numbers is an invariant. We start with the set containing 3 positive numbers. It is not possible to reach 6 positive numbers.

   Proof 2: When 2 numbers are multiplied by -1, the product of all the numbers does not change. Initially the product is -36. It is not possible to make it 36.

2. Prove that an 8×8 square cannot be covered by 11 straight tetrominoes and 5 L-tetrominoes.

   Color the board as shown in the figure below. Each straight tetromino covers either 0 or 2 or 4 gray squares, thus all 11 straight tetrominoes must cover an even number of gray squares. Each L-tetromino covers either 1 or 3 gray squares, thus 5 of them must cover an odd number of gray squares. Therefore all the tiles together must cover an odd number of gray squares. But our board contains 32 (which is even) gray squares. Thus the board cannot be covered by these 16 tiles.

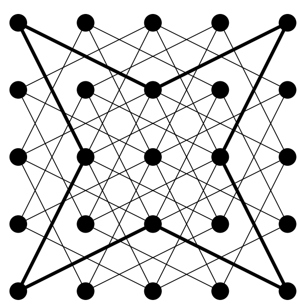


3. Solve for x: $|2x+3| - |x| = 3$.

   Case I. $2x+3 \geq 0$ *and* $x \geq 0$.
   $2x + 3 - x = 3$
   $x = 0$, *this root satisfies both condition* $2x+3 \geq 0$ *and* $x \geq 0$.
   Case II. $2x+3 < 0$ *and* $x \geq 0$.
   This says that $x < -1.5$ *and* $x \geq 0$ *which is impossible, so there are no solutions in this case.*
   Case III. $2x+3 \geq 0$ *and* $x < 0$.
   $2x + 3 + x = 3$
   $x = 0$, *doesn't satisfy the condition* $x < 0$.
   Case IV. $2x+3 < 0$ *and* $x < 0$.
   $-(2x+3) + x = 3$
   $x = -6$, *satisfies both conditions* $2x+3 < 0$ *and* $x < 0$.
   Answer: $x = 0$ *and* $x = -6$.

4. Show that there is no reentrant knight's tour on a 5×5 chessboard.

   Proof 1: In a reentrant knight's tour black and white squares must alternate. But a 5×5 *chessboard has 13 squares of one color and 12 squares of the other color, so it is not possible to have a cylce in which the colors alternate.*



   Proof 2: Draw a graph representing legal moves of a knight. Look at the corner vertices. They all have degree 2, thus, in order to visit the corner vertices, we must use both edges at each corner vertex. Those 8 edges form a cycle. It is not possible to add more edges to this cycle, but the cycle misses many points. Therefore there is no Hamilton cycle, and thus there is no reentrant tour.

- **Extra credit** (15 points): Twelve 1×1 cells of a 10×10 square are infected. Two cells are neighbors if they share at least one vertex (thus an inner cell has 8 neighbors). In one time unit, the cells with at least 4 infected neighbors become infected. Can the infection spread to the whole square?

  The idea is the same as that for the practice problem, only now 2 cells sharing just one vertex are also considered neighbors. For simplicity, draw more cells around the 10×10 *square so that each cell in our square has exactly 8 neighbors. Draw a graph in which vertices represent cells, and 2 vertices are connected if and only if the corresponding cells are neighbors. There will be 100 vertices representing 100 cells of our square, and more vertices around represending the cells around the square. Thus each of our 100 vertices will have degree 8. Now consider the number of edges with one infected endpoint and one uninfected endpoint. This is the analogue of the perimeter here. Initially the number of such edges is at most* $12 \times 8 = 96$. *When a vertex with 4 infected neighbors becomes infected, the number of such edges doesn't change. When a vertex with 5, 6, 7, or 8 infected neighbors becomes infected, then the number of such edges decreases. But this number never increases. If all the 100 vertices become infected, then the number of such edges becomes 116 which is larger than 96. Contradiction.*