MATH 145

Test 3 - Solutions

- $\int_{-4}^{2}|x+2|dx = \int_{-4}^{-2}|x+2|dx + \int_{-2}^{2}|x+2|dx = -\int_{-4}^{-2}(x+2)dx + \int_{-2}^{2}(x+2)dx = -\left(\frac{x^2}{2}+2x\right)\Big|_{-4}^{-2} + \left(\frac{x^2}{2}+2x\right)\Big|_{-2}^{2} = -(2-4)+(8-8)+(2+4)-(2-4) = 10$
  Note: another way to do this problem is to interpret the integral in terms of areas.

1. Two circles, S and T, and a point A are given. Find points B on S and C on T such that $\triangle ABC$ is isosceles with $AB = AC$, $\angle ABC = \angle ACB = 75°$, and $\angle BAC = 30°$. Assume that a solution exists.

   Rotate the circle S through an angle of $30°$ around A (toward circle T). Call the image S'. Let C be an intersection point of S' and T. Rotate the point C back - get a point B on the original circle S. Then $AB = AC$ and $\angle BAC = 30°$. The other angles are as required because $\angle ABC = \angle ACB$ since $\triangle ABC$ is isosceles, and the sum of all the angles in a triangle is $180°$.

2. Find integer numbers a and b such that $6 = 67a + 25b$.

   Use Euclid's algorithm to find $1 = 67 \cdot 3 - 25 \cdot 8$. Then $6 = 67 \cdot 18 - 25 \cdot 48$, thus $a = 18$ and $b = -48$ work. (Note: of course, this is not the only pair of such integers!)

3. Two players play the following game.
   - Turns alternate.
   - At each turn, a player removes either 1 or 2 counters from a pile that had initially 10 counters.
   - The game ends when all counters have been removed.
   - The player who takes the last counter loses.

   Find a winning strategy for one of the players.

   On our last turn we want to leave one counter. Then our opponent will have to take it, and they will lose. Notice that no matter how our opponent plays, we can always play in such a way that the number of counters our opponent takes plus the number of counters we take is equal to 3 (namely, if they take 1, we can take 2; if they take 2, we can take 1). Thus on our next to last turn we'll leave 4 (then no matter how they play, we'll be able to leave 1). On the turn before that we want to leave 7. This means that we should let our opponent go first. Then, if they take x, we take $3 - x$, and leave 7. Then we leave 4, then we leave 1, and we win.

4. The parabola $y = x^2 + 2$ has two tangent lines that pass through the origin. Find their equations.

   (Draw a picture so that you see what's going on.) Let the slope of such a tangent line be m, then its equation is $y = mx$. Let (a, ma) be the touching point. Since this point lies on the parabola, $ma = a^2 + 2$. The slope of the parabola at the touching point must be m, therefore $2a = m$. Substituting this into the first equation gives $2a^2 = a^2 + 2$. Then $a = \pm\sqrt{2}$, and $m = \pm 2\sqrt{2}$. Thus the equations of the tangent lines are $y = 2\sqrt{2}x$ and $y = -2\sqrt{2}x$.

- Two lines, p and q, and a point A are given. Find points B on p and C on q such that $\triangle ABC$ is isosceles with $AB = BC$, and $\angle ABC = 90°$. Assume that a solution exists.

  (Draw a solution.) Notice that $AC = \sqrt{2}AB$ and $\angle BAC = 45°$. This means that if we rotate the point B through $45°$ around A, let's call the image B', then C lies on the line AB' and $AC = \sqrt{2}AB'$. Thus to find such points, we have to rotate the line p through $45°$ around A, let's call the image p', and then draw a line p'' parallel to p' and such that the distance from A to p'' is $\sqrt{2}$ times the distance from A to p'. (To do this, pick any point X' on p', draw the line AX', find X'' on AX' such that $AX'' = \sqrt{2}AX'$, and draw the line p'' through X'' and parallel to p'.) Let C be the intersection point of p'' and q. Draw the line AC. Let B' be the intersection point of p' and AC. Rotate B' through $45°$ around A to get a point B on the original line p. Now we have $\angle BAC = 45°$ and $AC = \sqrt{2}AB' = \sqrt{2}AB$.