MATH 145 Final Exam - Solutions 14 December 2005

1. Prove that if 40 coins are distributed among 9 bags so that each bag contains at least one coin, then at least two bags contain the same number of coins. Is your proof direct, by contradiction, or by contrapositive?

   Suppose that the 9 bags contain different numbers of coins, and the total number of coins is 40. Let $a_1 < a_2 < \ldots < a_9$ be the numbers of coins in the 9 bags. Then $a_1 \geq 1$, $a_2 \geq 2$, ..., $a_9 \geq 9$, and $40 = a_1 + a_2 + \ldots + a_9 \geq 1 + 2 + \ldots + 9 = 45$. Since $40 < 45$, we get a contradiction, thus it is not possible for all bags to contain different numbers of coins. This is a proof by contradiction.

2. Find a formula for $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \ldots + \dfrac{1}{(2n-1)(2n+1)}$.

   Let $S_n = \dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \ldots + \dfrac{1}{(2n-1)(2n+1)}$. *Then*

   $S_1 = \dfrac{1}{1\cdot 3}$,

   $S_2 = \dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} = \dfrac{6}{15} = \dfrac{2}{5}$,

   $S_3 = \dfrac{2}{5} + \dfrac{1}{5\cdot 7} = \dfrac{15}{35} = \dfrac{3}{7}$,

   $S_4 = \dfrac{3}{7} + \dfrac{1}{7\cdot 9} = \dfrac{28}{63} = \dfrac{4}{9}$.

   We guess from these that $S_n = \dfrac{n}{2n+1}$.

   Proof by induction:

   Basis step. *If $n = 1$, the equality* $S_1 = \dfrac{1}{1\cdot 2\cdot 1 + 1}$ *holds.*

   Inductive step. *Suppse the formula* $S_n = \dfrac{n}{2n+1}$ *holds for some $n = k$, i.e.* $S_k = \dfrac{k}{2k+1}$.

   We want to prove that it holds for $n = k+1$, i.e. $S_{k+1} = \dfrac{k+1}{2(k+1)+1}$.

   $$S_{k+1} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = S_k + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3} = \frac{k+1}{2(k+1)+1}.$$

3. December 14, 2005 is a Wednesday. What day of the week is December 14, 2025?

   First notice that years 2008, 2012, 2016, 2020, and 2024 are leap years. The other 15 years from 2006 to 2025 (including these) are not. Since non-leap years contain 365 days and leap years contain 366 days, the number of days that pass between December 14, 2005 and December 14, 2025 is $15 \cdot 365 + 5 \cdot 366 \equiv 1 \cdot 1 + 5 \cdot 2 \equiv 11 \equiv 4 \ (mod\ 7)$, *therefore December 14, 2025 is 4 days after a Wednesday, i.e. is a Sunday.*

4. Solve the inequality: $x^2 - |7x + 15| \geq 3$.

   Consider the following two cases:

Case I. $7x+15 \geq 0$, *or* $x \geq -\frac{15}{7}$. *Then* $|7x+15| = 7x+15$, *and the inequality becomes*
$x^2-(7x+15) \geq 3$
$x^2-7x-15 \geq 3$
$x^2-7x-18 \geq 0$
$(x-9)(x+2) \geq 0$
$x \leq -2$ *or* $x \geq 9$
Combining this solution set with the condition $x \geq -\frac{15}{7}$ *gives* $[-\frac{15}{7}, 2] \cup [9, +\infty)$.
Case II. $7x+15 < 0$, *or* $x < -\frac{15}{7}$. *Then* $|7x+15| = -(7x+15)$, *and the inequality becomes*
$x^2+(7x+15) \geq 3$
$x^2+7x+12 \geq 0$
$(x+3)(x+4) \geq 0$
$x \leq -4$ *or* $x \geq -3$
Combining this solution set with the condition $x < -\frac{15}{7}$ *gives* $(-\infty, -4] \cup [-3, -\frac{15}{7}]$.
Finally, we take the union of all solution intervals from both cases, and we get: $(-\infty, -4] \cup [-3, 2] \cup [9, +\infty)$.

5. The number 8^{2005} is written on a blackboard (it contains over 1800 digits, so I won't write it out here). The sum of its digits is calculated, then the sum of the digits of the result is calculated and so on, until we get a single digit. What is this digit?

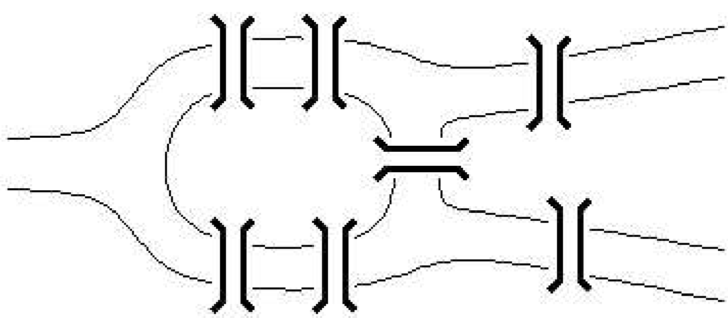
   We know that the sum of the digits of any natural number is congruent to the number itself modulo 9. Thus when we calculate the sum of the digits, the remainder modulo 9 does not change. It remains to calculate the remainder of 8^{2005} *modulo 9:* $8^{2005} \equiv (-1)^{2005} \equiv -1 \equiv 8 \ (mod\ 9)$. *Thus the single digit that remains at the end is* 8.

6. A box contains 300 matches. Players take turns removing no more than half the matches in the box. The player who cannot take any match(es) loses. Find a winning strategy for one of the players.

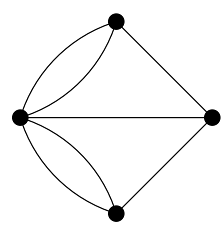
   If we want to win, we want to leave 1 match on our last turn so that our opponent loses. To be able to do this, we should get 2 matches on our last turn (if we get 3 or more, we won't be able to remove all but one). This means that we want to leave 3 the turn before that: our opponent can only remove 3 leaving us with 2. To be able to leave 3, we should get 4 or 5 or 6. Notice that if we leave 7 the turn before that, our opponent will be able to take 1 or 2 or 3 leaving us with 6 or 5 or 4 respectively which is exactly what we want. We can leave 7 if we get any number from 8 to 14 inclusive. Thus if we leave 15 before that our opponent will be able to take 1-7 leaving us with 14-8. And so on... Notice that the numbers of matches we should leave on our turn (going backwards) are 1, 3, 7, 15, - these all are one less than powers of 2. So it seems that the strategy is to always leave one less than a power of 2. Namely, we get a strategy for the first player: he/she should remove 45 matches leaving 255; the opponent will remove some number from 1 to 127 (inclusive), leaving some number between 254 and 128 (inclusive). The first player then will be able to remove the some number between 127 and 1 leaving 127 (which means that he/she removed no more than half); the opponent will remove some number between 1 and 63 leaving some number between 126 and 64. The first player should remove the required number between 63 and 1 to leave 63; the opponent will remove 1-31 leaving

62-32. The first player should remove the required number between 31 and 1 to leave 31; the opponent will remove 1-15 leaving 30-16. The first player should remove the required number between 15 and 1 to leave 15. Then, as described above, the first player should leave 7, 3, and, finally, 1, and he/she wins.

7. Below is a plan of Konigsberg. As discussed in class, it is not possible to design a tour of the town that crosses each bridge exactly once and returns to the starting point. Could the citizens of Konigsberg find such a tour by building a new bridge?



We will use a graph to represent the city as follows. Let each of the four pieces of land be represented by a vertex, and let each bridge be represented by an edge connecting the corresponding vertices. Then we get the following graph:



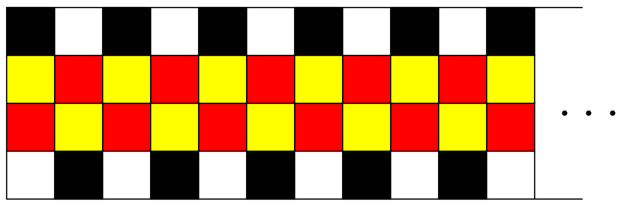
Existance of a tour described in the problem is equiavalent to existance of an Euler cycle in this graph. We know that an Euler cycle exists if and only if the degree of each vertex is even. However, the degrees of the vertices in our graph are 5, 3, 3, and 3, i.e. all odd, and this is how we know that no such tour exists. Now, if we add one more bridge we will change the degrees of two vertices (corresponding to the pieces of land that this bridge connects), thus the degrees of two vertices will become even, but the degrees of the other two vertices will remain odd. Therefore no such tour will exist even if one bridge is built.

8. Evaluate the integral: $\displaystyle\int_0^1 \arcsin(x)dx$ (hint: use areas).

Since $\arcsin(x)$ *(defined on* $[-1,1]$*) is the inverse function of* $\sin(x)$ *on* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$*, their graphs are symmetric about the line* $y = x$*, thus the value of* $\displaystyle\int_0^1 \arcsin(x)dx$*, the area of the region under the graph of* $\arcsin(x)$ *on* $[0,1]$*, is equal to the area of the region between* $y = \sin(x)$ *and* $y = 1$ *from* $x = 0$ *to* $x = \frac{\pi}{2}$*. (Draw the graphs to see this!) The* *latter area can be calculated by* $\int_0^{\frac{\pi}{2}} (1 - \sin(x))dx = (x + \cos(x)) \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1.$

- Is it possible for a chess knight to pass through all the squares of a 4×2005 board having visited each square exactly once, and return to the initial square?

  The answer is no. This can be proved by contradiction, namely, let's assume that such a reentrant knight tour exists. Color the board as shown on the picture below:

  Notice that from any black square a knight can only get to a yellow square; from any white square a knight can only get to a red square. Since there are 2005 squares of each color, the tour must contain 2005 pairs "black, yellow" and 2005 pairs "white, red". However, there is no way to get from a yellow square to a white one or from a red square to a black one. We get a contradiction.