Practice Test 1 - Solutions

- What does "a and b are relatively prime" mean?

  Integers a and b are called relatively prime if their greatest common divisor is 1. (Note: actually, the test will only cover Logic, Types of Proofs, Induction, and Dirichlet's Principle, so the question will be on one of these topics.)

1. Let $P(x, y)$ denote the proposition "$x \leq y$" where x and y are real numbers. Determine the truth values of
   (a) $\forall x P(-x, x)$ *is false. For example, if $x = -1$ then $-x = 1$. Then $P(1, -1)$ says that $1 \leq -1$ which is false. So $P(-x, x)$ is not true for all x.*
   (b) $\exists x \exists y P(x, y)$ *is true. Example: $x = 1$, $y = 2$. Then $P(1, 2)$ is true since $1 \leq 2$.*
   (c) $\forall x \exists y P(x, y)$ *is true. For any x, let $y = x + 1$. Then $P(x, x + 1)$ is true since $x \leq x + 1$.*
   (d) $\exists x \forall y P(x, y)$ *is false. No matter what x is, consider $y = x - 1$. Then $P(x, x - 1)$ (meaning $x \leq x - 1$) is false. So no matter what x is, not all y will satisfy $P(x, y)$.*
   (e) $\forall x \forall y P(x, y)$ *is false. Counterexample: $x = 2$, $y = 1$. Then $P(2, 1)$ is false, so not all pairs x, y satisfy $P(x, y)$.*

2. Prove that for any integers n and m, if $nm + 2n + 2m$ is odd then both n and m are odd (you may only use the definitions of even and odd numbers; do not use any properties unless you prove them). Is your proof direct, by contrapositive, or by contradiction?

   This statement can be proved using different approaches.

   A proof by contrapositive: Assume that not both n and m are odd. Then at least one of them is even.
   If n is even then $n = 2a$ for some integer a, then $nm + 2n + 2m = 2am + 2 \cdot 2a + 2m = 2(am + 2a + m)$ is even.
   If m is even then $m = 2b$ for some integer b, then $nm + 2n + 2m = 2nb + 2n + 2 \cdot 2b = 2(nb + n + 2b)$ is even.

   A proof by contradiction: Suppose that $nm + 2n + 2m$ is odd but not both n and m are odd.
   Since $nm + 2n + 2m$ is odd, $nm + 2n + 2m = 2k + 1$ for some integer k. Then $nm = 2k + 1 - 2n - 2m = 2(k - n - m) + 1$ is odd. But if at least one of n, m is even then their product is even (see the proof above). We get a contradiction. Therefore both n and m are odd.

   (Note: This type of proof by contradiction is based on the following principle.
   The implication we want to prove is of the form $p \rightarrow q$. It is equivalent to $\neg p \vee q$. We assume $p \wedge \neg q$ and get a contradiction. Therefore $p \wedge \neg q$ is false, so $\neg(p \wedge \neg q)$ is true. But $\neg(p \wedge \neg q)$ is equivalent to $\neg p \vee q$.)

3. Prove that for any natural n,

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof by Mathematical Induction.

Basis step. If $n = 1$ *the identity is* $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$ *which is true.*

Inductive step. Assume the identity holds for $n = k$*, i.e.*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

We want to prove that it holds for $n = k+1$*, i.e.*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}.$$

Using the inductive hypothesis (the identity for $n = k$*) we have:*

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) + (k+1)(k+2) =$$

$$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} =$$
$$\frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} = \frac{(k+1)(k+2)[k+3]}{3}.$$

4. Kevin is paid every other week on Friday. Show that every year, in some month he is paid three times.

Every year contains at least 365 *days (leap years contain* 366 *days). Since* $\frac{365}{7} = 52 + \frac{1}{7}$*, every year contains at least* 52 *weeks. Kevin is paid every other week, so he is paid at least* $\frac{52}{2} = 26$ *times a year. Next we will use Dirichlet's box principle (months will be "boxes" and paydays will be "objects"). Since there are* 12 *months and* 26 *(more than* $2 \cdot 12$*) paydays, at least one month contains at least three paydays. Thus in some month Kevin is paid three times.*

- Let f be a one-to-one function from $X = \{1, 2, \ldots, n\}$ onto X. Let $f^k = \underbrace{f \circ f \circ \ldots \circ f}_{k\ times}$ denote the k-fold composition of f with itself. Show that for some positive integer m, $f^m(x) = x$ for all $x \in X$.

  For every power f^k *of* f*, consider the* n*-tuple of its values:* $\{f^k(1), f^k(2), \ldots, f^k(n)\}$*. Since* f *is one-to-one, its power* f^k *is one-to-one, therefore the* n*-tuple shown above is a permutation of the set* $\{1, 2, \ldots, n\}$*. There are finitely many permutations (namely, there are* $n!$ *of them), thus by Dirichlet's box principle some powers of* f *are equal. (For example, consider the first* $n! + 1$ *powers of* f*. Since there are more powers than possible permutations, Dirichlet's box principle applies.) Thus for some* $i > j$ *we have* $f^i(x) = f^j(x)$ *for all* $x \in X$*. Finally,* f *has an inverse because it is one-to-one from a finite set onto itself, and thus is a bijection, so we can apply* f^{-j} *to both sides of* $f^i(x) = f^j(x)$*. Then* $f^{i-j}(x) = x$ *for all* $x \in X$*.*