MATH 145 — Test 1 - Solutions

- If $P(x)$ is a propositional function, which of the following are logically equivalent: $\neg\exists x P(x)$, $\exists x \neg P(x)$, $\forall x \neg P(x)$? Answer: *$\neg\exists x P(x)$ and $\forall x \neg P(x)$ are logically equivalent.*

1. Let $P(x, y)$ denote the propositional function "$xy = 0$" where x and y are real numbers. Determine the truth values of the following propositions. (Provide reasons!)
   (a) *$\exists x \forall y P(x, y)$ is true. For example, if $x = 0$ then for any y we have $0y = 0$.*
   (b) *$\forall x \exists y P(x, y)$ is true. For any x we can choose $y = 0$ and then $x0 = 0$.*
   (c) *$\forall x \forall y P(x, y)$ is false. Counterexample: $x = 1$, $y = 1$. Then $1 \cdot 1 \neq 0$.*
   (d) *$\exists! x \forall y P(x, y)$ is true. Only $x = 0$ satisfies the condition that for any y, $xy = 0$. Because if $x \neq 0$ then e.g. for $y = 1$ we'll have $xy \neq 0$.*
   (e) *$\forall x \exists! y P(x, y)$ is false. For $x = 0$ there are infinitely many values of y that give $xy = 0$ (in fact, y can be any real number). Thus not for any x such a y is unique.*

2. Prove that if a is rational and b is irrational then $a + b$ is irrational.
   Is your proof direct, by contradiction, or by contrapositive?
   Proof by contradiction: Suppose a is rational, b is irrational, and yet $a + b$ is not irrational, i.e. rational. Then $a = \dfrac{c}{d}$ for some integers c, $d \neq 0$, and $a + b = \dfrac{f}{g}$ for some integers f, $g \neq 0$. Then $b = (a + b) - a = \dfrac{f}{g} - \dfrac{c}{d} = \dfrac{fd - cg}{gd}$ is rational (notice that $gd \neq 0$). We get a contradiction to the hypothesis that b is irrational. Thus the statement is true.

3. Use Mathematical Induction to prove that for any natural n, $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$.
   Basis step: if $n = 1$ then the identity is $\dfrac{1}{2} = 1 - \dfrac{1}{2}$ which is true.
   Inductive step: assume the identity holds for $n = k$, i.e. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^k} = 1 - \dfrac{1}{2^k}$.
   We want to prove that the identity holds for $n = k + 1$, i.e. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^{k+1}} = 1 - \dfrac{1}{2^{k+1}}$.
   Using the inductive hypothesis we have: $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^{k+1}} = \left(\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots + \dfrac{1}{2^k}\right) + \dfrac{1}{2^{k+1}} = 1 - \dfrac{1}{2^k} + \dfrac{1}{2^{k+1}} = 1 - \dfrac{2}{2^{k+1}} + \dfrac{1}{2^{k+1}} = 1 - \dfrac{1}{2^{k+1}}$.

4. Ten points are chosen randomly inside a 3×3 square. Prove that there are two of them with distance at most $\sqrt{2}$.
   Divide the square into nine 1×1 squares. Since there are 9 squares and 10 points, by Dirichlet's Box Principle at least one square contains at least two points. Since the length of a diagonal of the unit square is equal to $\sqrt{2}$, the distance between two such points is at most $\sqrt{2}$.

- Prove that for any integer number $n \geq 3$, $\left(1 + \dfrac{1}{n}\right)^n < n$.

  Proof by Mathematical Induction. Basis step: if $n = 3$ we have $\left(1 + \dfrac{1}{3}\right)^3 = \left(\dfrac{4}{3}\right)^4 = \dfrac{64}{27} < 3$.

  Inductive step: assume the inequality holds for $n = k$, i.e. $\left(1 + \dfrac{1}{k}\right)^k < k$.

  We want to show that the inequality holds for $n = k + 1$, i.e. $\left(1 + \dfrac{1}{k+1}\right)^{k+1} < k + 1$.

  The inductive hypothesis implies that $\left(\dfrac{k+1}{k}\right)^k < k$. Since $\dfrac{k+2}{k+1} < \dfrac{k+1}{k}$ (because $(k + 2)k = k^2 + 2k < k^2 + 2k + 1 = (k + 1)^2$), we have $\left(1 + \dfrac{1}{k+1}\right)^{k+1} = \left(\dfrac{k+2}{k+1}\right)^{k+1} = \left(\dfrac{k+2}{k+1}\right)^k \left(\dfrac{k+2}{k+1}\right) < \left(\dfrac{k+1}{k}\right)^k \left(\dfrac{k+2}{k+1}\right) < k \cdot \dfrac{k+2}{k+1} = \dfrac{k(k+2)}{k+1} < \dfrac{(k+1)^2}{k+1} = k + 1$.