Name: ________________________________

**Answer the question** (5 points):

- If \( P(x) \) is a propositional function, which of the following are logically equivalent: 
  \[ \neg \exists x P(x), \, \exists x \neg P(x), \, \forall x \neg P(x) \] ?

  Answer: ______________________________________________________________________

**and do any 3 of the following problems** (15 points each):

1. Let \( P(x, y) \) denote the propositional function \( "xy = 0" \) where \( x \) and \( y \) are real numbers. Determine the truth values of the following propositions. (Provide reasons!)
   - \( \exists x \forall y P(x, y) \),
   - \( \forall x \exists y P(x, y) \),
   - \( \forall x \forall y P(x, y) \),
   - \( \exists x \exists y P(x, y) \),
   - \( \forall x \exists ! y P(x, y) \),
   - \( \forall x \forall ! y P(x, y) \).

2. Prove that if \( a \) is rational and \( b \) is irrational then \( a + b \) is irrational. Is your proof direct, by contradiction, or by contrapositive?

3. Use Mathematical Induction to prove that for any natural \( n \),
   \[ \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \]

4. Ten points are chosen randomly inside a \( 3 \times 3 \) square. Prove that there are two of them with distance at most \( \sqrt{2} \).

**For extra credit** (15 points):

- Prove that for any integer number \( n \geq 3 \),
  \[ \left( 1 + \frac{1}{n} \right)^n < n. \]
1. Let $P(x, y)$ denote the propositional function “$xy = 0$” where $x$ and $y$ are real numbers. Determine the truth values of the following propositions. (Provide reasons!)

(a) $\exists x \forall y P(x, y)$

(b) $\forall x \exists y P(x, y)$

(c) $\forall x \forall y P(x, y)$

(d) $\exists ! x \forall y P(x, y)$

(e) $\forall x \exists ! y P(x, y)$
2. Prove that if $a$ is rational and $b$ is irrational then $a + b$ is irrational.

Is your proof direct, by contradiction, or by contrapositive?
3. Use Mathematical Induction to prove that for any natural $n$,

$$
\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}.
$$
4. Ten points are chosen randomly inside a $3 \times 3$ square. Prove that there are two of them with distance at most $\sqrt{2}$. 
For extra credit: Prove that for any integer number $n \geq 3$, \( \left( 1 + \frac{1}{n} \right)^n < n \).