MATH 145

Test 2 - Solutions

• Is it true or false that if $a$ and $b$ are not integers then $\frac{a}{b}$ is irrational?

Answer: False (counterexample: $\frac{1}{2} = \frac{1}{2}$ is a rational number)

1. Solve for $x$: $(x - 3)^{x^2 - 8x + 15} = 0$

The power is $0$ iff $x - 3 = 0$ and $x^2 - 8x + 15 \neq 0$ (because $0^0$ is undefined), however, this system has no solutions.

Note: it was supposed to be $(x - 3)^{x^2 - 8x + 15} = 1$, so if you solved this equation, it was accepted.

Case I. $x - 3 = 1$, then $x = 4$.

Case II. $x^2 - 8x + 15 = 0$, $x - 3 \neq 0$. The first equation holds when $x = 3$ or $x = 5$, but since $x - 3 \neq 0$, we have only one solution in this case: $x = 5$.

Case III. $x - 3 = -1$, $x^2 - 8x + 15$ is even. However, the only solution of the first equation is $x = 2$, and for this value $x^2 - 8x + 15$ is not even. Therefore there are no solutions in this case.

Answer: $x = 4$, $x = 5$.

2. Find a formula for

$$
\prod_{i=1}^{2n-1} \left( 1 - \frac{(-1)^i}{i} \right) = \left( 1 - \frac{-1}{1} \right) \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{-1}{3} \right) \cdots \left( 1 - \frac{-1}{2n-1} \right)
$$

and prove it.

Let $A_n = \prod_{i=1}^{2n-1} \left( 1 - \frac{(-1)^i}{i} \right)$, then

$A_1 = 1 - \frac{-1}{1} = 2$

$A_2 = \left( 1 - \frac{-1}{1} \right) \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{-1}{3} \right) = 2 \cdot \frac{1}{2} \cdot \frac{4}{3} = \frac{4}{3}$

$A_3 = \left( 1 - \frac{-1}{1} \right) \left( 1 - \frac{1}{2} \right) \left( 1 - \frac{-1}{3} \right) \left( 1 - \frac{1}{4} \right) \left( 1 - \frac{-1}{5} \right) = 2 \cdot \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} = \frac{6}{5}$

Guess: $A_n = \frac{2n}{2n-1}$.

Proof by Mathematical Induction:

Basis step. If $n = 1$, we have $A_1 = 2 = \frac{2}{1}$ as shown above.
Inductive step. Suppose $A_k = \frac{2k}{2k-1}$.

We want to prove that $A_{k+1} = \frac{2(k+1)}{2(k+1)-1} = \frac{2k+2}{2k+1}$.

Using the inductive hypothesis, we have:

$$A_{k+1} = \left(1 - \frac{-1}{1}\right) \left(1 - \frac{-1}{2}\right) \left(1 - \frac{-1}{3}\right) \cdots \left(1 - \frac{-1}{2k-1}\right) =$$

$$\left(1 - \frac{-1}{1}\right) \left(1 - \frac{-1}{2}\right) \left(1 - \frac{-1}{3}\right) \cdots \left(1 - \frac{-1}{2k-1}\right) \left(1 - \frac{1}{2k}\right) \left(1 - \frac{-1}{2k+1}\right) =$$

$$A_k \left(1 - \frac{-1}{2k}\right) \left(1 - \frac{-1}{2k+1}\right) = \frac{2k}{2k-1} \cdot \frac{2k-1}{2k} \cdot \frac{2k+2}{2k+1} = \frac{2k+2}{2k+1}.$$

3. Start with the set $\{1, 4, 32, 128, 256\}$. In each step, you may divide one number by 2 and multiply another number by 2. Is it possible to reach the set $\{512, 32, 16, 16, 2\}$?

When we divide one number by 2 and multiply another number by 2, we do not change the product of all five numbers. Thus the product is an invariant. However, the product of the numbers in the set $\{512, 32, 16, 16, 2\}$ is not equal to the product of the numbers in the initial set $\{1, 4, 32, 128, 256\}$.

(This can be shown in different ways, e.g.):

$$1 \cdot 4 \cdot 32 \cdot 128 \cdot 256 = 2^0 \cdot 2^2 \cdot 2^5 \cdot 2^7 \cdot 2^8 = 2^{0+2+5+7+8} = 2^{22},$$

$$512 \cdot 32 \cdot 16 \cdot 16 \cdot 2 = 2^9 \cdot 2^5 \cdot 2^4 \cdot 2^4 \cdot 2^1 = 2^{9+5+4+4+1} = 2^{23},$$

$$\left(\text{or} \frac{1 \cdot 4 \cdot 32 \cdot 128 \cdot 256}{512 \cdot 32 \cdot 16 \cdot 16 \cdot 2} = \frac{1 \cdot 4 \cdot 32 \cdot 128 \cdot 256}{2 \cdot 16 \cdot 16 \cdot 32 \cdot 512} = \frac{1}{2} \cdot \frac{4}{16} \cdot \frac{32}{16} \cdot \frac{128}{32} \cdot \frac{256}{512} = \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{2} \neq 1.\right)$$

Therefore it is not possible to reach the set $\{512, 32, 16, 16, 2\}$.

4. Prove that an $8 \times 8$ board cannot be covered by 7 T-tetrominoes and 9 L-tetrominoes.

![Tetrominoes](image-url)

- **straight tetromino**
- **T-tetromino**
- **square tetromino**
- **L-tetromino**
- **skew tetromino**

Color the board using the traditional chessboard coloring. Then there are 32 black squares and 32 white squares. Each T-tetromino covers either 1 or 3 black squares, therefore seven T-tetrominoes must cover an odd number of black squares. Each L-tetromino covers 2 black squares, therefore nine L-tetrominoes must cover 18 (i.e. and odd number of) black squares. Thus all sixteen tiles together must cover an (odd + even=) odd number of black squares. However, the board has an even number (32) of black squares, therefore a covering is not possible.