MATH 145

Test 2 - Solutions

- Is it true or false that if a and b are not integers then $\dfrac{a}{b}$ is irrational?

  Answer: *False (counterexample: $\dfrac{1/2}{1} = \dfrac{1}{2}$ is a rational number)*

1. Solve for x: $(x-3)^{x^2-8x+15} = 0$

   The power is 0 iff $x - 3 = 0$ and $x^2 - 8x + 15 \neq 0$ (because 0^0 is undefined), however, this system has no solutions.

   Note: *it was supposed to be $(x-3)^{x^2-8x+15} = 1$, so if you solved this equation, it was accepted.*

   Case I. $x - 3 = 1$, *then* $x = 4$.
   Case II. $x^2 - 8x + 15 = 0$, $x - 3 \neq 0$. *The first equation holds when $x = 3$ or $x = 5$, but since $x - 3 \neq 0$, we have only one solution in this case: $x = 5$.*
   Case III. $x - 3 = -1$, $x^2 - 8x + 15$ *is even. However, the only solution of the first equation is $x = 2$, and for this value $x^2 - 8x + 15$ is not even. Therefore there are no solutions in this case.*

   Answer: $x = 4$, $x = 5$.

2. Find a formula for

$$\prod_{i=1}^{2n-1}\left(1 - \frac{(-1)^i}{i}\right) = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\cdots\left(1 - \frac{-1}{2n-1}\right)$$

   and prove it.

   Let $A_n = \prod_{i=1}^{2n-1}\left(1 - \frac{(-1)^i}{i}\right)$, *then*

$$A_1 = 1 - \frac{-1}{1} = 2$$

$$A_2 = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right) = 2 \cdot \frac{1}{2} \cdot \frac{4}{3} = \frac{4}{3}$$

$$A_3 = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{-1}{5}\right) = 2 \cdot \frac{1}{2} \cdot \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{6}{5} = \frac{6}{5}$$

   Guess: $A_n = \dfrac{2n}{2n-1}$.
   Proof by Mathematical Induction:
   Basis step. If $n = 1$, we have $A_1 = 2 = \dfrac{2}{1}$ as shown above.

Inductive step. Suppose $A_k = \dfrac{2k}{2k-1}$.
We want to prove that $A_{k+1} = \dfrac{2(k+1)}{2(k+1)-1} = \dfrac{2k+2}{2k+1}$.
Using the inductive hypothesis, we have:

$$A_{k+1} = \left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\cdots\left(1 - \frac{-1}{2(k+1)-1}\right) =$$

$$\left(1 - \frac{-1}{1}\right)\left(1 - \frac{1}{2}\right)\left(1 - \frac{-1}{3}\right)\cdots\left(1 - \frac{-1}{2k-1}\right)\left(1 - \frac{1}{2k}\right)\left(1 - \frac{-1}{2k+1}\right) =$$

$$A_k\left(1 - \frac{1}{2k}\right)\left(1 - \frac{-1}{2k+1}\right) = \frac{2k}{2k-1}\cdot\frac{2k-1}{2k}\cdot\frac{2k+2}{2k+1} = \frac{2k+2}{2k+1}.$$

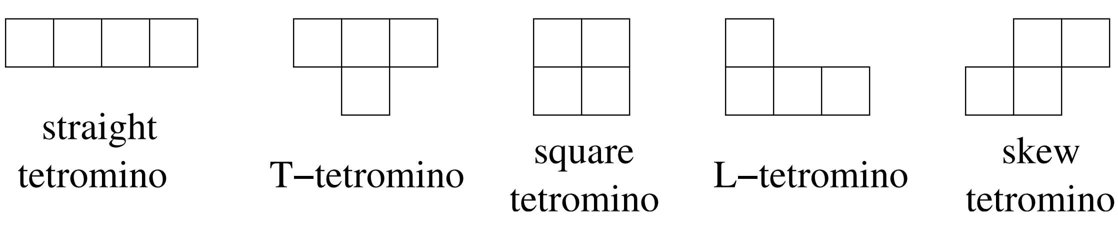
3. Start with the set $\{1, 4, 32, 128, 256\}$. In each step, you may divide one number by 2 and multiply another number by 2. Is it possible to reach the set $\{512, 32, 16, 16, 2\}$?

   When we divide one number by 2 *and multiply another number by* 2, *we do not change the product of all five numbers. Thus the product is an invariant. However, the product of the numbers in the set* $\{512, 32, 16, 16, 2\}$ *is not equal to the product of the numbers in the initial set* $\{1, 4, 32, 128, 256\}$.
   (This can be shown in different ways, e.g.:)
   $1 \cdot 4 \cdot 32 \cdot 128 \cdot 256 = 2^0 \cdot 2^2 \cdot 2^5 \cdot 2^7 \cdot 2^8 = 2^{0+2+5+7+8} = 2^{22}$,
   $512 \cdot 32 \cdot 16 \cdot 16 \cdot 2 = 2^9 \cdot 2^5 \cdot 2^4 \cdot 2^4 \cdot 2^1 = 2^{9+5+4+4+1} = 2^{23}$;

   (or $\dfrac{1 \cdot 4 \cdot 32 \cdot 128 \cdot 256}{512 \cdot 32 \cdot 16 \cdot 16 \cdot 2} = \dfrac{1 \cdot 4 \cdot 32 \cdot 128 \cdot 256}{2 \cdot 16 \cdot 16 \cdot 32 \cdot 512} = \dfrac{1}{2}\cdot\dfrac{4}{16}\cdot\dfrac{32}{16}\cdot\dfrac{128}{32}\cdot\dfrac{256}{512} = \dfrac{1}{2}\cdot\dfrac{1}{4}\cdot 2 \cdot 4 \cdot \dfrac{1}{2} = \dfrac{1}{2} \neq 1$.*)*

   Therefore it is not possible to reach the set $\{512, 32, 16, 16, 2\}$.

4. Prove that an 8×8 board cannot be covered by 7 T-tetrominoes and 9 L-tetrominoes.



   Color the board using the traditional chessboard coloring. Then there are 32 *black squares and* 32 *white squares. Each T-tetromino covers either* 1 *or* 3 *black squares, therefore seven T-tetrominoes must cover an odd number of black squares. Each L-tetromino covers* 2 *black squares, therefore nine L-tetrominoes must cover* 18 *(i.e. and odd number of) black squares. Thus all sixteen tiles together must cover an (odd + even=) odd number of black squares. However, the board has an even number (32) of black squares, therefore a covering is not possible.*