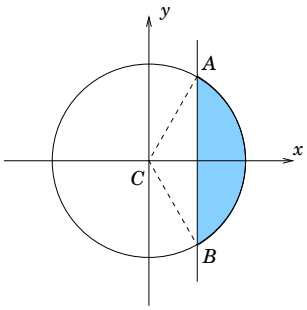
MATH 145
Test 3 - Solutions

- Is the following statement true of false: if $\int_a^b f(x)dx > 0$ then $f(x) \geq 0$ for all $x \in [a, b]$.

  Answer: *False* (*Counterexample:* $f(x) = \sin(x)$, $a = 0$, $b = \frac{3\pi}{2}$)

1. Sketch the region $S = \{(x, y) \mid x \geq 2, x^2 + y^2 \leq 16\}$ and find its area.

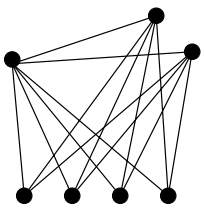
   The region consists of all points on or inside the circle given by $x^2 + y^2 = 16$ *and on or to the right of the vertical line* $x = 2$. *It is shown below (shaded):*



   The area of the region is the difference between the area of sector ACB *and the area of triangle* ACB. *Since* $CD = 2 = \frac{1}{2}CA$, $\angle ACD = 60°$, *therefore* $\angle ACB = 120°$, *and the area of the sector is* $\frac{1}{3}$ *of the area of the circle, i.e.* $\frac{1}{3}\pi 4^2 = \frac{16\pi}{3}$. *The area of triangle* ACB *is* $\frac{1}{2}|CD| \cdot |AB| = \frac{1}{2} \cdot 2 \cdot 4\sqrt{3} = 4\sqrt{3}$. *Thus the area of the shaded region is* $\frac{16\pi}{3} - 4\sqrt{3}$.

2. A graph $K_{k,l,m}$ has $k + l + m$ vertices divided into three sets: k vertices in one set, l vertices in another set, and m vertices in the third set. Two vertices are connected if and only if they are in different sets. Does $K_{1,2,4}$ have an Euler cycle?

   A graph $K_{1,2,4}$ *is shown below. It has one vertex of degree* 6, *two vertices of degree* 5, *and four vertices of degree* 3. *Since a graph has an Euler cycle if and only if all vertices have even degrees, this graph does not have an Euler cycle.*

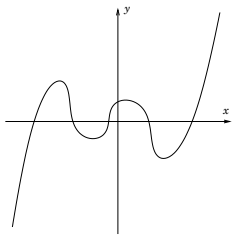


3. Two players play the following game.
   - Turns alternate.
   - At each turn, a player removes 1, 2, or 4 counters from a pile that had initially 10 counters.
   - The game ends when all counters have been removed.
   - The player who takes the last counter wins.

   Find a winning strategy for one of the players.

   In order to win we want to take the last counter. Thus we will win if we get 1, 2, *or* 4 *counters on our last turn. This means that if we leave* 3 *counters on the turn before that, our opponent will be able to take* 1 *or* 2 *counters leaving us with* 2 *or* 1 *respectively. In order to be able to leave* 3, *we want to get* 4, 5, *or* 7 *counters. Now notice that if we leave* 6 *on the turn before that, our opponent will be able to take* 1, 2, *or* 4, *leaving us with* 5, 4, *or* 2 *respectively, and all of these are winning positions for us. Thus we want to go first and take* 4 *counters on our first turn. This will leave* 6. *Then, as described above, we will win. (This is a strategy for the first player.)*

4. Explain why the curve shown below cannot be the graph of a cubic polynomial.



   Reason 1: a cubic polynomial has at most 3 *real roots while the given curve has* 5 *x-intercepts.*
   Reason 2: a cubic poynomial has at most 2 *local extremum points (because its derivative is a quadratic polynomial and thus has at most* 2 *real roots) while the give curve has* 4 *local extrema.*

- **Extra credit** (15 points): What is the ratio of the 5-dimensional volume of a 5-dimensional ball to the 4-dimensional volume of its boundary (the analog of the surface area)?

  The 5*-dimensional volume of a* 5*-dimensional ball is proportional to the* 5*-th degree of the radius. Let* $V = cr^5$ *where* c *is a constant. The* 4*-dimensional volume of its boundary (let's denote it by* B*) is the derivative of the volume:* $B = 5cr^4$. *Therefore the ratio is* $\dfrac{V}{B} = \dfrac{cr^5}{5cr^4} = \dfrac{r}{5}$.