Rational Trigonometric Sum was a Zip-Line problem linked to the article "Values of Trigonometric Functions" by Jeffrey Bergen in the February issue. The article gave a characterization by elementary methods of all rational angles α (angles that are π times a rational number) such that both $\sin^2(\alpha)$ and $\cos^2(\alpha)$ are rational numbers. **Problem 226** asked which rational angles α make the single quantity $\sin(\alpha) + \cos(\alpha)$ a rational number.

Correct solutions were given by students Michael Abram and Santhosh Karnik (Wheeler High School, GA), as well as by the Armstrong Problem Solvers and the Northwestern University Problem Solving Group. There was one incorrect solution. Three of the solutions used the fact that $(\sin(\alpha) + \cos(\alpha))^2 = 1 + \sin(2\alpha)$. As the group from Northwestern explains, if $\sin(\alpha) + \cos(\alpha)$ is rational, so is its square $(\sin(\alpha) + \cos(\alpha))^2$, and so $\sin(2\alpha)$ must be rational. By the first theorem in Bergen's article, this is enough to limit the value of $\sin(2\alpha)$ to one of the values –1, –1/2, 0, 1/2, and 1. The values –1/2, 1/2, and 1 can be ruled out because they make $\sqrt{(1 + \sin(2\alpha))}$ irrational. The only permissible values of $\sin(2\alpha)$ are 0 and –1, which occur just when $\alpha = \pi k/2$ and $\alpha = 3\pi/4 + \pi k$ for any integer k.

"FIVE MORE MINUTES, KIDS!"

In **Glimpsing a Heart,** Jeffrey Liebner posed a probability question. You are playing a game where two cards are dealt face down to each player. As the cards are being dealt and before you see your own cards, you catch a glimpse of the cards of the opponent sitting to your right: all you can conclude is that at least one of the cards is a heart. **Problem 227** asked whether that glimpse changes the probability that your opponent has two aces.

The Armstrong Problem Solvers, the Bethel College Problem-Solving Group, Taylor A, the Skidmore College Problem Group, and student David Montgomery (Westmont College) all arrived at the same two probabilities and the conclusion that, yes, the probability does change. A variety of methods were used to compute the two probabilities; what follows is David's approach. Before the glimpse, since there are $C(4,2) = 6$ unordered pairs of aces out of $C(52,2) = 1326$ possible pairs of cards, the probability that the opponent has two aces is 6/1326 = 1/221. (Here $C(4,2)$ represents four cards taken two at a time, or "4 choose 2.") After the glimpse, only 3 of the original 6 pairs of two aces are possible. Also, the total number of possible pairs is reduced by $C(39, 2) = 741$, the number of pairs containing no heart, leaving 1326–741 = 585 pairs with at least one heart. This gives a new probability of 3/585 = 1/195, which is greater than 1/221.

The Skidmore group approached the problem using conditional probability. If A is the event that the opponent has two aces and H is the event that the opponent has at least one heart, then the problem is asking if $P(A)$ and $P(A|H)$ differ. The latter quantity is a conditional probability, the probability that A occurs given that H occurs; it can be computed using the formula $P(A|H) = P(A \cap H) / P(H)$, where $A \cap H$ is the event that the opponent has both two aces and at least one heart. After computing the probabilities $P(A \cap H)=1/442$ and $P(H)=15/34$, the group also arrived at $P(A)=1/221$ and $P(A|H)=1/195$.

However, all of the solutions made one implicit assumption: that all pairs of cards containing at least one heart are *equally likely* to be glimpsed. Given the admittedly imprecise statement of the problem, that might not be the case. Suppose, for instance, that you have a better chance of identifying the suit of the *first* of the two cards being dealt to your opponent, a reasonable scenario if your opponent is taking a quick single peek at both of her cards together on the table. One version of this situation would have a heart identifiable during the glimpse if and only if it appeared on the first of the two cards dealt. In this case, if you recompute $P(A \cap H)$ and $P(H)$ above you might just find that the probabilities $P(A)$ and $P(A|H)$ don't differ by much, if at all!

CLEANING UP

Here's an example of three points in $\mathbb{R}^2$ such that no pair of the points is related by $\leq$: (1,4), (2,3), and (3,2). Good luck with Problem 232!

Also, Michael Abram correctly solved Problem 222, but his solution was not acknowledged in the April issue.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. All problems and/or solutions may be submitted to Derek Smith, Mathematics Department, Lafayette College, Easton, PA 18042. Electronic submissions (preferred) may also be sent to `smithder@lafayette.edu`. Please include your name, email address, school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is November 10, 2009.

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