PROBLEMS

PROPOSALS

To be considered for publication, solutions should be received by November 1, 2009.

1821. *Proposed by Abdullah Al-Sharif and Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.*

Let $ABCD$ be a convex quadrilateral, let X and Y be the midpoints of sides BC and DA respectively, and let O be the point of intersection of diagonals of $ABCD$. Prove that O lies inside of quadrilateral $ABXY$ if and only if

$$\text{Area}\,(AOB) < \text{Area}\,(COD).$$

1822. *Proposed by Pham Van Thuan, Hanoi University of Science, Hanoi, Vietnam.*

Let u and v be positive real numbers. Prove that

$$\frac{1}{8}\left(17 - \frac{2uv}{u^2+v^2}\right) \le \sqrt[3]{\frac{u}{v}} + \sqrt[3]{\frac{v}{u}} \le \sqrt{(u+v)\left(\frac{1}{u}+\frac{1}{v}\right)}.$$

Find conditions under which equality holds.

1823. *Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY.*

Let n and k be positive integers. Find a closed-form expression for the number of permutations of $\{1, 2, \ldots, n\}$ for which the initial k entries have the same parity, but the initial $k+1$ entries do not. (As an example, for the permutation 5712463, the number of initial entries of the same parity is 3, the order of the set $\{5, 7, 1\}$.)

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. Each solution should begin on a separate sheet.

Solutions and new proposals should be mailed to Elgin Johnston, Problems Editor, Department of Mathematics, Iowa State University, Ames IA 50011, or mailed electronically (ideally as a LaTeX file) to ehjohnst@iastate.edu. All communications should include the reader's name, full address, and an e-mail address and/or FAX number on every page.

1824. *Proposed by Cezar Lupu, student, University of Bucharest, Bucharest, Romania.*

Let f be a continuous real-valued function defined on $[0, 1]$ and satisfying

$$\int_0^1 f(x)\,dx = \int_0^1 xf(x)\,dx.$$

Prove that there exists a real number c, $0 < c < 1$, such that

$$cf(c) = \int_0^c xf(x)\,dx.$$

1825. *Proposed by Greg Oman and Kevin Schoenecker, The Ohio State University, Columbus, OH.*

Let R be a ring with more than two elements. Prove that there exist subsets S and T of R, both closed under multiplication, and such that $S \not\subseteq T$ and $T \not\subseteq S$. (Note: We do not assume that R is commutative nor do we assume that R has a multiplicative identity.)

Quickies

Answers to the Quickies are on page 232.

Q991. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let f be a real-valued, differentiable function on $[a, b]$ with $f'(x) \geq f(x) > 0$ for all $x \in [a, b]$. Prove that

$$\int_a^b \frac{1}{f(x)}\,dx \leq \frac{1}{f(a)} - \frac{1}{f(b)}.$$

Q992. *Proposed by Luis H. Gallardo, University of Brest, Brest, France.*

Let n be a perfect number. Prove that if $n - 1$ and $n + 1$ are both prime, then $n = 6$.

Solutions

Staying closer to the center **June 2008**

1796. *Proposed by Matthew McMullen, Otterbein College, Westerville, OH.*

A point is selected at random from the region inside of a regular n-gon. What is the probability that the point is closer to the center of the n-gon than it is to the n-gon itself?

Solution by Houghton College Problem Solving Group, Houghton College, Houghton, NY.

Consider the regular n-gon centered at the origin and with the midpoint of one side at the point $(1, 0)$ in polar coordinates. The desired probability is the same as the probability that a point inside the triangle with vertices $O = (0, 0)$, $A = (1, 0)$ and $B = \left(\sec\frac{\pi}{n}, \frac{\pi}{n}\right)$ is closer to $(0, 0)$ than it is to side $\overline{AB}$. Now let (r, θ) be a point inside of the triangle and equidistant from O and $\overline{AB}$. Then

$$r = 1 - r\cos\theta, \qquad \text{from which} \qquad r = \frac{1}{1 + \cos\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}.$$

The area of the region inside of the triangle and between this curve and the origin is

$$\frac{1}{2}\int_0^{\pi/n} r^2\,d\theta = \frac{1}{8}\int_0^{\pi/n} \sec^4\frac{\theta}{2}\,d\theta = \frac{1}{12}\tan\frac{\pi}{2n}\left(\tan^2\frac{\pi}{2n}+3\right).$$

Dividing this result by $\frac{1}{2}\tan\frac{\pi}{n}$, the area of the triangle, gives the desired probability,

$$\frac{1}{12}\left(4-\sec^4\frac{\pi}{2n}\right).$$

Also solved by Robert A. Agnew, Michael Andreoli, Herb Bailey, Thomas Bass and Kenneth Massey and Alden Starnes, Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Albania), Brian Bradie, Bruce S. Burdick, Robert Calcaterra, John Christopher, Chip Curtis, Jim Delany, Fejéntaláltuka Szeged Problem Group (Hungary), John N. Fitch, Natacha Fontes-Merz and Ramiro Fontes, Leon Gerber, Tom Gearhart, Marvin Glover, G.R.A.20 Problem Solving Group (Italy), Jeffrey M. Groah, Jerrold W. Grossman, Lee O. Hagglund, Eugene A. Herman, Michael Hitchman, Andrew Incognito, Eugen J. Ionascu, Victor Y. Kutsenok, Elias Lampakis (Greece), Charles Lindsey, David Lovit, Bob Mallison, Tim McDevitt, Kim McInturff, Missouri State University Problem Solving Group, Ronald G. Mosier, Erik Murphy and James Bush, Gail Nord, Gary L. Raduns, Edward Schmeichel, Allen Schwenk, Albert Stadler (Switzerland), Britton Stamper, James Swenson, Marian Tetiva (Romania), Bob Tomper, Michael Vowe (Switzerland), John B. Zacharias, and the proposer. There was one solution with no name and two incorrect submissions.

Limit of a radical sum **June 2008**

1797. *Proposed by Ovidiu Furdui, The University of Toledo, Toledo, OH.*

Let a, b, and c be nonnegative real numbers. Find the value of

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{n^2+kn+a}}{\sqrt{n^2+kn+b}\sqrt{n^2+kn+c}}.$$

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

The value of the limit is $2(\sqrt{2}-1)$.

The kth term of the sum can be rewritten as

$$t_{n,k} = \frac{\sqrt{n^2+kn+a}}{\sqrt{n^2+kn+b}\sqrt{n^2+kn+c}} = \frac{1}{n}\cdot\frac{\sqrt{1+\frac{k}{n}+\frac{a}{n^2}}}{\sqrt{1+\frac{k}{n}+\frac{b}{n^2}}\sqrt{1+\frac{k}{n}+\frac{c}{n^2}}}.$$

Given $\epsilon > 0$ there exists N so that for $n > N$, each of a/n^2, b/n^2, and c/n^2 is less than ϵ. For such n,

$$\frac{1}{n}\cdot\frac{\sqrt{1+\frac{k}{n}}}{1+\frac{k}{n}+\epsilon} \le t_{n,k} \le \frac{1}{n}\cdot\frac{\sqrt{1+\frac{k}{n}+\epsilon}}{1+\frac{k}{n}}.$$

The sum for $k = 1$ to n for the expressions on both sides of this double inequality are Riemann sums, respectively, for the following two integrals:

$$I_L(\epsilon) = \int_0^1 \frac{\sqrt{1+x}}{1+x+\epsilon}\,dx \longrightarrow \int_0^1 \frac{\sqrt{1+x}}{1+x}\,dx = 2(\sqrt{2}-1),$$

$$I_R(\epsilon) = \int_0^1 \frac{\sqrt{1+x+\epsilon}}{1+x}\,dx \longrightarrow \int_0^1 \frac{\sqrt{1+x}}{1+x}\,dx = 2(\sqrt{2}-1),$$

where the limits are justified because the integrands converge uniformly on $[0, 1]$ as $\epsilon \to 0$. Hence the desired limit is $2(\sqrt{2}-1)$ as claimed.

Also solved by Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Albania), David M. Bradley, Brian Bradie, Robert Calcaterra, Elliot Cohen (France), Knut Dale (Norway), Manuel Fernández-López (Spain), John N. Fitch, G.R.A.20 Problem Solving Group (Italy), Eugene A. Herman, Andrew Incognito, Eugen J. Ionascu, Victor Y. Kutsenok, Elias Lampakis (Greece), David Lovit, Bob Mallison, Reiner Martin, Missouri State University Problem Solving Group, Ronald G. Mosier, Paolo Perfetti (Italy), Éric Pité (France), Gabriel T. Prăjitură, Allen Schwenk, C. R. Selvaraj and Suguna Selvaraj, Nicholas C. Singer, Albert Stadler (Switzerland), Bob Tomper, Michael Vowe (Switzerland), John B. Zacharias, and the proposer.

Minimum of a radical sum **June 2008**

1798. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let x, y, and z be positive real numbers with $x+y+z=xyz$. Find the minimum value of

$$\sqrt{1+x^2}+\sqrt{1+y^2}+\sqrt{1+z^2},$$

and find all (x, y, z) for which the minimum occurs.

Solution by Michael Reid, University of Central Florida, Orlando, FL.

The minimum value is 6, which occurs if and only if $x=y=z=\sqrt{3}$. Let $\alpha = \operatorname{Arctan} x$, $\beta = \operatorname{Arctan} y$, and $\gamma = \operatorname{Arctan} z$. Then

$$\tan\gamma = z = \frac{x+y}{xy-1} = \tan(-(\alpha+\beta)),$$

so γ differs from $-(\alpha+\beta)$ by a multiple of π. Because $\alpha, \beta, \gamma \in (0, \pi/2)$, we have $0 < \alpha+\beta+\gamma < 3\pi/2$, and it follows that $\alpha+\beta+\gamma = \pi$.

In terms of α, β, γ, the quantity to be minimized is $\sec\alpha + \sec\beta + \sec\gamma$. The function $f(t) = \sec t$ is strictly convex on the interval $(0, \pi/2)$, because $f''(t) = \sec t\,(2\tan^2 t + 1)$ is positive on the interval. Therefore we have

$$\sec\alpha + \sec\beta + \sec\gamma \geq 3\sec\left(\frac{\alpha+\beta+\gamma}{3}\right) = 3\sec\left(\frac{\pi}{3}\right) = 6,$$

with equality if and only if $\alpha=\beta=\gamma=\pi/3$. In terms of x, y, and z, this condition is equivalent to $x=y=z=\sqrt{3}$.

Also solved by George Apostolopoulos (Greece), Herb Bailey, Michel Bataille (France), Mihaly Bencze (Romania), D. Bennett and H. To, Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Albania), Brian Bradie, Krista Buchheit, Bruce S. Burdick, Robert Calcaterra, Hongwei Chen, Chip Curtis, Knut Dale (Norway), Charles R. Diminnie, Fejéntaláltuka Szeged Problem Group (Hungary), John Ferdinands, Micheal Goldenberg and Mark Kaplan, Peter Gressis and Dennis Gressis, John G. Heuver, Eugen J. Ionascu, D. Kipp Johnson, Hwan-jin Kim (Korea), Elias Lampakis (Greece), Kee-Wai Lau (China), Jizhou Li, David Lovit, Phil McCartney, Tadele Mengesha, Ronald G. Mosier, Evangelos Mouroukos (Greece), Ken'ichi Nagasaki (Japan), Paolo Perfetti (Italy), Gabriel T. Prăjitură, Toufic Saad, C. R. Selvaraj and Suguna Selvaraj, Nicholas C. Singer, Albert Stadler (Switzerland), Marian Tetiva (Romania), Nora S. Thornber, George Tsapakidis (Greece), Zhexiu Tu, University of Central Oklahoma Problem Solving Group, Michael Vowe (Switzerland), John B. Zacharias, and the proposer. There was one solution with no name and six incorrect submissions.

Matrix matters **June 2008**

1799. *Proposed by Luz DeAlba, Drake University, Des Moines, IA.*

Let $s_1, s_2, \ldots, s_n$ be real numbers with $0 < s_1 < s_2 < \cdots < s_n$. For $1 \leq i \leq j \leq n$ define $a_{ij} = a_{ji} = s_j$, and let A be the $n \times n$ matrix $A = [a_{ij}]_{1\leq i,j\leq n}$.

(a) Calculate $\det A$.

(b) Let $A^{-1} = [b_{ij}]_{1\leq i,j\leq n}$. Find the value of $\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}$.

Solution by Vadim Ponomarenko, San Diego State University, San Diego, CA.

(a) Let T be the matrix with -1 for each entry on the (first) super diagonal and 0 for all other entries. Then

$$(I+T)A(I+T^{\text{tr}}) = \operatorname{diag}(s_1 - s_2, s_2 - s_3, \dots, s_{n-1} - s_n, s_n).$$

Because $\det(I+T) = 1$, it follows that

$$\det A = \det\left((I+T)A(I+T^{\text{tr}})\right) = (s_1 - s_2)(s_2 - s_3)\cdots(s_{n-1} - s_n)s_n.$$

(b) Let

$$B = \left((I+T)A(I+T^{\text{tr}})\right)^{-1} = \operatorname{diag}\left(\frac{1}{s_1 - s_2}, \frac{1}{s_2 - s_3}, \dots, \frac{1}{s_{n-1} - s_n}, \frac{1}{s_n}\right).$$

Hence

$$A^{-1} = (I+T^{\text{tr}})B(I+T) = B + BT + T^{\text{tr}}B + T^{\text{tr}}BT.$$

If we let

$$\alpha = \frac{1}{s_1 - s_2} + \frac{1}{s_2 - s_3} + \cdots + \frac{1}{s_{n-1} - s_n},$$

then we find that the sum of the entries of B is $\alpha + 1/s_n$, the sum of the entries of BT is $-\alpha$, the sum of the entries of $T^{\text{tr}}B$ is $-\alpha$, and the sum of the entries of $T^{\text{tr}}BT$ is α. It follows that the sum of the entries of A^{-1} is $1/s_n$.

Also solved by Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Albania), Brian Bradie, Bruce S. Burdick, Kevin Byrnes, Robert Calcaterra, Minh Can, Hongwei Chen, Elliot Cohen (France), Chip Curtis, Knut Dale (Norway), Fejéntaláltuka Szeged Problem Group (Hungary), John Ferdinands, Manuel Fernández-López (Spain), Micheal Goldenberg and Mark Kaplan, Eugene A. Herman, M. Hako and K. Sanderson and H. To, Victor Y. Kutsenok, Reiner Martin, Missouri State University Problem Solving Group, Éric Pité (France), Gabriel T. Prăjitură, Rob Pratt, Michael Reid, Edward Schmeichel, C. R. Selvaraj and Suguna Selvaraj, Raul A. Simon (Chile), Nicholas C. Singer, John H. Smith, Albert Stadler (Switzerland), James Swenson, Marian Tetiva, Dave Trautman, Michael Vowe (Switzerland), John B. Zacharias, and the proposer.

Minimizing a ratio of areas **June 2008**

1800. *Proposed by Michel Bataille, Rouen, France.*

Let ABC be a triangle, let E be a fixed point on the interior of side AC, and let F be a fixed point on the interior of side AB. For P on $\overline{EF}$, define

$$\rho(P) = \frac{[PBC]^2}{[PCA][PAB]}.$$

For which P does $\rho(P)$ take on its minimum value? What is this minimal value?

Solution by Michael Vowe, Therwil, Switzerland.

We determine the normalized barycentric (or *areal*) coordinates for E, F, and P with respect to $A = (1,0,0)$, $B = (0,1,0)$, and $C = (0,0,1)$. Because E is on the interior of AC and F is on the interior of AB, we have

$$E = uA + (1-u)C = (u, 0, 1-u) \quad \text{and} \quad F = vA + (1-v)B = (v, 1-v, 0),$$

where $0 < u, v < 1$. Because P must be on the interior of EF, we have

$$P = xE + (1-x)F = (xu + (1-x)v, (1-x)(1-v), x(1-u)),$$

with $0 < x < 1$. The area of triangle PBC (as a fraction of the area of ABC) is just the first coordinate of P, so $[PBC] = xu + (1-x)v$. Similarly,

$$[PCA] = (1-x)(1-v) \qquad \text{and} \qquad [PAB] = x(1-u).$$

Hence,

$$\rho(P) = \frac{(xu + (1-x)v)^2}{x(1-x)(1-u)(1-v)}.$$

By the arithmetic-geometric mean inequality we obtain

$$\rho(P) \geq \frac{\left(2\sqrt{xu(1-x)v}\right)^2}{x(1-x)(1-u)(1-v)} = \frac{4uv}{(1-u)(1-v)},$$

with equality if and only if $xu = (1-x)v$, that is, if and only if $x = v/(v+u)$. Therefore the minimum value of $\rho(P)$ is $\frac{4uv}{(1-u)(1-v)}$ and occurs at the point

$$P_m = \left(\frac{2uv}{u+v}, \frac{u(1-v)}{u+v}, \frac{v(1-u)}{u+v}\right).$$

In other words, P_m is the point on EF with $\frac{EP}{FP} = \frac{u}{v} = \frac{(AB)(CE)}{(AC)(BF)}$, and the minimum value is $4\frac{(BE)(CF)}{(AE)(AF)}$.

Also solved by Herb Bailey, Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Albania), Robert Calcaterra, Chip Curtis, Michael Goldenberg and Mark Kaplan, Peter Gressis and Dennis Gressis, Eugen J. Ionascu, L. R. King, Victor Y. Kutsenok, Elias Lampakis (Greece), Ken'ichi Nagasaki (Japan), Gabriel T. Prăjitură, Joel Schlosberg, Raul A. Simon (Chile), Albert Stadler (Switzerland), John B. Zacharias, and the proposer.

Answers

Solutions to the Quickies from page 228.

A991. Because $f'(x) \geq f(x)$ on $[a, b]$, it follows that $\left(e^{-x} f(x)\right)' \geq 0$, and hence that $e^{-x} f(x)$ is positive and nondecreasing on $[a, b]$. Thus

$$\frac{f(x)}{e^x} \geq \frac{f(a)}{e^a}, \quad \text{from which,} \quad \frac{1}{f(x)} \leq \frac{e^a}{f(a)e^x}$$

on $[a, b]$. Therefore

$$\int_a^b \frac{dx}{f(x)} \leq \int_a^b \frac{e^a}{f(a)e^x}\, dx = -\frac{e^a}{f(a)}\left(\frac{1}{e^b} - \frac{1}{e^a}\right) = \frac{1}{f(a)}\left(1 - \frac{e^a}{e^b}\right). \tag{1}$$

However

$$\frac{f(b)}{e^b} \geq \frac{f(a)}{e^a}, \quad \text{so it follows that} \quad \frac{f(a)}{f(b)} \leq \frac{e^a}{e^b}.$$

It then follows from (1) that

$$\int_a^b \frac{dx}{f(x)} \leq \frac{1}{f(a)}\left(1 - \frac{e^a}{e^b}\right) \leq \frac{1}{f(a)}\left(1 - \frac{f(a)}{f(b)}\right) = \frac{1}{f(a)} - \frac{1}{f(b)}.$$

This completes the proof.

A992. It is clear that n must be even, so $n = 2^{p-1}(2^p - 1)$, where p and $2^p - 1$ are prime. It is easy to check that if $n > 6$, then $p > 2$ and $n \equiv 1 \pmod 9$. Thus, if $n > 6$ then $n - 1$ is a multiple of 9 and hence is not prime. Note that the conditions of the problem are satisfied for the perfect number $n = 6$.