PROBLEMS

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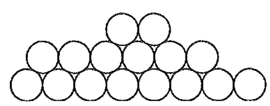
PROPOSALS

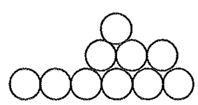
To be considered for publication, solutions should be received by March 1, 2010.

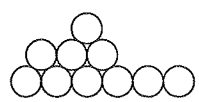
1826. *Proposed by Michael Woltermann, Washington & Jefferson College, Washington, PA.*

A block fountain of coins is an arrangement of n identical coins in rows such that the coins in the first row form a contiguous block, and each row above that forms a contiguous block. As an example,

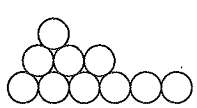


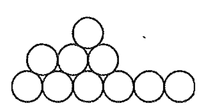
If a_n denotes the number of block fountains with exactly n coins in the base, then $a_n = F_{2n-1}$, where F_k denotes the kth Fibonacci number. (Wilf, generatingfunctionology, 1994.) How many block fountains are there if two fountains that are mirror images of each other are considered to be the same? That is, if two fountains such as





are the same, while two fountains such as





are different?

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. Each solution should begin on a separate sheet.

Solutions and new proposals should be mailed to Bernardo M. Abrego, Problems Editor-Elect, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St., Northridge CA 91330-8313, or mailed electronically (ideally as a LaTeX file) to mathmagproblems@csun.edu. All communications should include the reader's name, full address, and an e-mail address and/or FAX number.

1827. *Proposed by Christopher Hilliar, Texas A & M University, College Station, TX.*

Let A be an $n \times n$ matrix with integer entries and such that each column of A is a permutation of the first column. Prove that if the entries in the first column do not sum to 0, then this sum divides $\det(A)$.

1828. *Proposed by Stephen J. Herschkorn, Department of Statistics, Rutgers University, New Brunswick, NJ.*

Let α_0 be the smallest value of α for which there exists a positive constant C such that

$$\prod_{k=1}^{n} \frac{2k}{2k-1} \leq Cn^{\alpha}$$

for all positive integer n.

a. Find the value of α_0.

b. Prove that the sequence

$$\left\{ \frac{1}{n^{\alpha_0}} \prod_{k=1}^{n} \frac{2k}{2k-1} \right\}_{n=1}^{\infty}$$

is decreasing and find its limit.

1829. *Proposed by Oleh Faynshteyn, Leipzig, Germany.*

Let ABC be a triangle with $BC = a$, $CA = b$, and $AB = c$. Let r_a denote the radius of the excircle tangent to BC, r_b the radius of the excircle tangent to CA, and r_c the radius of the excircle tangent to AB. Prove that

$$\frac{r_a r_b}{(a+b)^2} + \frac{r_b r_c}{(b+c)^2} + \frac{r_c r_a}{(c+a)^2} \geq \frac{9}{16}.$$

1830. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let α and β be positive real numbers and let r be a positive rational number. Find necessary and sufficient conditions to ensure that there exist infinitely many positive integers m such that

$$\frac{\lfloor m\alpha \rfloor}{\lfloor m\beta \rfloor} = r.$$

Quickies

Answers to the Quickies are on page 316.

Q993. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let (X, d) be a metric space and let

$$A = \{f : f \text{ is a real valued nonexpansive function on } X\}.$$

Let x_0 and y_0 be two given points in X. Find

$$\sup\{|f(x_0) - f(y_0)| : f \in A\},$$

and justify your answer. (We say $f : X \to R$ is nonexpansive if $|f(x) - f(y)| \leq d(x, y)$ for all x and y in X.)

Q994. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let k be a positive real number. Find the value of

$$\int_0^1 \int_0^1 \left\{ \left(\frac{1}{x}\right)^k - \left(\frac{1}{y}\right)^k \right\} dx\, dy,$$

where $\{z\} = z - \lfloor z \rfloor$ denotes the fractional part of z.

Solutions

An unmatched tournament **October 2008**

2001. *Proposed by José H. Nieto, Universidad del Zulia, Maracaibo, Venezuela.*

A chess club has n members. Each member of the club has played against all but k of the other members. The club decides to hold a tournament in which each member plays exactly one game against those he/she has not played before. The tournament is played in rounds, with each player playing at most one game each round. Each round is scheduled by randomly selecting pairs who have not previously played against each other (and who are not already scheduled for the round) until no more such pairs are available for the round. Determine the maximum possible number of rounds for such a tournament. (For example, if the club has six members A, B, C, D, E, and F and the pairs that have never played against each other are AB, AC, AD, BC, BE, CF, DE, DF, and EF, then the tournament could consist of the following rounds:

$$1: \ AB, CF, DE, \qquad 2: \ BC, DF \qquad 3: \ AC, EF \qquad 4: \ AD, BE.)$$

Solution by Allen Schwenk, Western Michigan University, Kalamazoo, MI.

We first show that at most $2k-1$ rounds are needed. We model the situation with a regular graph of of n vertices and degree k. The vertices represent the players, and two vertices are connected by an edge if and only if the two corresponding players have not played. Now assume $2k-1$ rounds have been played and that two players, x and y, have not yet played. Because xy is an edge and each vertex is of order k, each of x and y could have been assigned to play in at most $k-1$ rounds, so there can be at most $2k-2$ rounds in which the xy pairing was not possible. Thus there must be at least one round of the $2k-1$ rounds in which neither x nor y is assigned to play. However, this is impossible because for each round, pairings are assigned until there are no remaining pairs who have not played.

We now show that there are situations in which $2k-1$ rounds are necessary. Suppose we have $n = 2k+2$ players (or vertices) labeled x, y, u_i, v_i for $1 \le i \le k$ and that the pairs that have not played (e.g., edges) are xu_i, u_iv_j, v_iy for $1 \le i,\ j \le k$, and $i \ne j$. It is easy to check that every vertex has degree k as required. Now assume that the first $k-1$ rounds are scheduled with k matches in each round as follows:

> Reading subscripts modulo k, round m, $1 \le m \le k-1$ consists of matches u_iv_{i+m}, $1 \le i \le k$.

For each round the same two players, x and y, are unmatched, and because they will not play each other, neither can be included in these first $k-1$ rounds. Thus after these $k-1$ rounds are completed, x still has to play k games. This means the tournament will last at least $2k-1$ rounds.

Also solved by Robert B. Eggleton, and the proposer.

An inequality **October 2008**

2002. *Proposed by Dorin Marghidanu, Colegiul National "A. I. Cuza," Corabia, Romania.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\cdots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1}\geq\frac{a_1+a_2+\cdots+a_n}{2}.$$

I. *Solution by Robert L. Doucette, McNeese State University, Lake Charles, LA.*
Cauchy's inequality for real numbers may be written

$$(x_1y_1+x_2y_2+\cdots+x_ny_n)^2\leq(x_1^2+x_2^2+\cdots+x_n^2)(y_1^2+y_2^2+\cdots+y_n^2).$$

Setting $x_i=\sqrt{a_i+a_{i+1}}$ and $y_i=a_i/\sqrt{a_i+a_{i+1}}$, for $1\leq i\leq n$, yields (with $a_{n+1}=a_1$)

$$\begin{aligned}&(a_1+a_2+\cdots+a_n)^2\\&\quad\leq 2(a_1+a_2+\cdots+a_n)\left(\frac{a_1^2}{a_1+a_2}+\frac{a_2^2}{a_2+a_3}+\cdots+\frac{a_{n-1}^2}{a_{n-1}+a_n}+\frac{a_n^2}{a_n+a_1}\right)\end{aligned}$$

This inequality is equivalent to the desired result.

II. *Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.*
Let $a_{n+1}=a_1$ and consider the two sums

$$S=\sum_{k=1}^{n}\frac{a_k^2}{a_k+a_{k+1}}\qquad\text{and}\qquad T=\sum_{k=1}^{n}\frac{a_{k+1}^2}{a_k+a_{k+1}}.$$

Then

$$S-T=\sum_{k=1}^{n}\frac{(a_k^2-a_{k+1}^2)}{a_k+a_{k+1}}=\sum_{k=1}^{n}(a_k-a_{k+1})=0,$$

so $S=T$. On the other hand, by the quadratic mean-arithmetic mean inequality,

$$\frac{a_k^2+a_{k+1}^2}{a_k+a_{k+1}}\geq\frac{a_k+a_{k+1}}{2},$$

so

$$S+T=\sum_{k=1}^{n}\frac{a_k^2+a_{k+1}^2}{a_k+a_{k+1}}\geq\sum_{k=1}^{n}\frac{a_k+a_{k+1}}{2}=\sum_{k=1}^{n}a_k.$$

Because $S=T$, this becomes

$$S\geq\frac{1}{2}\sum_{k=1}^{n}a_k,$$

which is the desired inequality.

Also solved by George Apostolopoulos (Greece), Byoung Tae Bae (Spain), Michel Bataille (France), Brain Bradie, Robert Calcaterra, Minh Can, Hongwei Chen, Chip Curtis, Daniele Degiorgi (Switzerland), Roger B. Eggleton, Fejéntaláltuka Szeged Problem Solving Group (Hungary), E. S. Friedkin, Peter Haggstrom (Australia), Eugene A. Herman, John G. Huever (Canada), Bianca-Teodora Iordache (Romania), D. Kipp Johnson, Lucyna Kabza, Omran Kouba (Syria), Victor Y. Kutsenok, Harris Kwong, Elias Lampakis (Greece), David P. Lang, *Kee-Wai Lau, (China), Peter W. Lindstrom, Graham Lord, Junaid N. Mansuri, Kim McInturff, Edward Omey (Belgium), Paolo Perfetti (Italy), Éric Pité (France), Ángel Plaza (Spain), Gabriel T. Prăjitură, Sebastián García Sáenz (Chile), Edward Schmeichel, Armend Shabani (Republic of Kosova), Joel M. Siegel, John Simons (Netherlands), Nicholas C. Singer, Albert Stadler (Switzerland), John Sumner and Aida Kadic-Galeb, Marian Tetiva (Romania), George Tsapakidis (Greece), Zhexiu Tu, Michael Vowe (Switzerland), Winona State Problem Solvers, and the proposer. There were two incorrect submissions.*

A convex set of fixed points **October 2008**

2003. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let $(X, \langle\,\rangle)$ be a real inner product space, and let

$$B = \{x \in X : \|x\| \leq 1\}$$

be the unit ball in X, where $\|x\| = \sqrt{\langle x, x\rangle}$. Let $f : B \to B$ be a function satisfying $\|f(x) - f(y)\| \leq \|x - y\|$ for all $x, y \in B$. Prove that the set of fixed points of f is convex.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.

Let a, b be distinct fixed points of f, and let c be a point on the line segment joining a and b different from either endpoint. We show that $f(c) = c$. We use the full version of the triangle inequality:

> For all $x, y \in X$, $\|x + y\| \leq \|x\| + \|y\|$ with equality if and only if one of x, y is a nonnegative multiple of the other.

Since $c = ta + (1 - t)b$ for some $t \in (0, 1)$, $a - c$ is a positive multiple $((1 - t)/t)$ of $c - b$. Hence

$$\begin{aligned}\|a - b\| &= \|(a - f(c)) + (f(c) - b)\| \leq \|a - f(c)\| + \|f(c) - b\| \\ &= \|f(a) - f(c)\| + \|f(c) - f(b)\| \leq \|a - c\| + \|c - b\| = \|a - b\|\end{aligned}$$

Since the second of the above inequalities is an equality, we have

$$\|a - f(c)\| = \|a - c\|, \quad \|f(c) - b\| = \|c - b\| \tag{1}$$

Because the first of the inequalities is an equality, we conclude from the triangle inequality that one of $a - f(c)$, $f(c) - b$ is a nonnegative multiple of the other. Since (1) shows that neither of these vectors is zero, there exists $\alpha > 0$ such that $a - f(c) = \alpha(f(c) - b)$, and so $f(c) = ua + (1 - u)b$, where $u = 1/(\alpha + 1) \in (0, 1)$. Substituting this expression for $f(c)$ and $ta + (1 - t)b$ for c in the second of equations (1) yields $u = t$ and therefore $f(c) = c$.

Also solved by Michel Bataille (France), Paul Budney, Bruce S. Burdick, Robert Calcaterra, Hongwei Chen, Chip Curtis, Jim Delany, Charles R. Diminnie, Robert L. Doucette, Fejéntaláltuka Szeged Problem Solving Group (Hungary), Dmitry Fleischman, G.R.A.20 Problem Solving Group (Italy), Cody Guinan and Jennifer Pajda, Dan Jurca, Omran Kouba (Syria), Elias Lampakis (Greece), Missouri State University Problem Solving Group, Éric Pité (France), Edward Schmeichel, Nicholas C. Singer, Albert Stadler (Switzerland), John Sumner and Aida Kadic-Galeb, Marian Tetiva (Romania), Bob Tomper, Haohao Wang and Jerzy Wojdylo, and the proposer.

A generator for F_{q^n} **October 2008**

2004. *Proposed by Jody M. Lockhart and William P. Wardlaw, U.S. Naval Academy, Annapolis, MD.*

Let A be an $n \times n$ matrix over the finite field F_q of q elements, and assume that A has multiplicative order $\operatorname{ord}(A) = q^n - 1$. Prove or give a counterexample to the following statement:

> A is a cyclic generator for F_{q^n}, that is, $\{0, A, A^2, \ldots, A^{q^n-1}\}$ is the finite field F_{q^n}.

Solution by Jim Delany, Emeritus, California Polytechnic State University, San Luis Obispo, CA.

Let $S = \{0, A, A^2, \dots, A^{q^n-1}\}$. The nonzero elements of S form a multiplicative (abelian) group, and the distributive law is obvious, so we must show that S is an additive (abelian) group.

Let $p(x)$ be the minimal polynomial of A and d be the degree of $p(x)$. Note that $d \le n$ since $p(x)$ is a factor of $\det(xI - A)$, the characteristic polynomial of A. Let

$$T = \{a_0 I + a_1 A + \cdots + a_{d-1}A^{d-1} : a_i \in F_q\}.$$

Then T is a vector space of dimension d over F_q, and hence is an additive group. Note also that T has q^d elements and that $q^d \le q^n$.

On the other hand $S \subset T$. To prove this, let $A^k \in S$ and write $x^k = p(x)q(x) + r(x)$ with $\deg(r(x)) < \deg(p(x)) = d$. Setting $x = A$ in this expression yields

$$A^k = p(A)q(A) + r(A) = 0 \cdot q(A) + r(A) = r(A) \in T.$$

Thus $q^n = |S| \le |T| = q^d \le q^n$ so $|S| = |T|$ and $S = T$. Hence S is a group under addition.

Also solved by Michel Bataille (France), Robert Calcaterra, Fejéntaláltuka Szeged Problem Solving Group (Hungary), Elias Lampakis (Greece), Éric Pité (France), Nicholas C. Singer, Gregory P. Wene, and the proposers.

A comparison test **October 2008**

2005. *Proposed by Ovidiu Furdui, Campia Turzii, Cluj, Romania.*

Let $f : [0, \infty) \to (0, \infty)$ be an increasing, differentiable function with continuous derivative, and let k be a nonnegative integer. Prove that

$$\int_0^\infty \frac{x^k}{f(x)}\,dx \quad \text{converges if and only if} \quad \int_0^\infty \frac{x^k}{f(x) + f'(x)}\,dx \quad \text{converges.}$$

Solution by Nicholas C. Singer, Annandale, VA.

Since $f'(x) \ge 0$ everywhere, the "only if" part is trivial. Suppose the second integral converges. Because f is a positive continuous function on $[0, 1]$, it attains a positive minimum there. Thus $\int_0^1 (x^k/f(x))\,dx$ is finite and can be neglected. For $X > 1$ and nonnegative integer k,

$$\int_1^X \frac{x^k dx}{f(x)} = \int_1^X \frac{x^k dx}{f(x) + f'(x)} + \int_1^X \frac{x^k f'(x) dx}{f(x)(f(x) + f'(x))}$$

$$\int_1^X \frac{dx}{f(x)} \le \int_1^X \frac{dx}{f(x) + f'(x)} + \int_1^X \frac{f'(x)\,dx}{f^2(x)}$$

$$\le \int_1^X \frac{dx}{f(x) + f'(x)} + \frac{1}{f(1)}.$$

The right hand side is bounded as $X \to \infty$, so the claim is true for $k = 0$. Suppose the claim is true for nonnegative integer k and suppose $\int_1^\infty x^{k+1}dx/(f(x) + f'(x))$ converges. The integrand dominates $x^k/(f(x) + f'(x))$, so $\int_1^\infty x^k dx/(f(x) + f'(x))$

converges. Then by the induction hypothesis, $\int_1^\infty x^k dx/f(x)$ converges. Now

$$\begin{aligned}\int_1^X \frac{x^{k+1}dx}{f(x)} &\le \int_1^X \frac{x^{k+1}dx}{f(x)+f'(x)} + \int_1^X \frac{x^{k+1}f'(x)\,dx}{f^2(x)} \\ &= \int_1^X \frac{x^{k+1}dx}{f(x)+f'(x)} - \frac{x^{k+1}}{f(x)}\Big|_1^X + \int_1^X \frac{(k+1)x^k dx}{f(x)} \\ &\le \int_1^X \frac{x^{k+1}dx}{f(x)+f'(x)} + \frac{1}{f(1)} + \int_1^X \frac{(k+1)x^k dx}{f(x)}.\end{aligned}$$

The terms on the right are bounded as $X \to \infty$, thus the claim is true for $k+1$ and we've proven the result by induction.

Also solved by Michel Bataille (France), Hongwei Chen, E. S. Friedkin, Kee-Wai Lau (China), Peter W. Lindstrom, José Miguel Pacheco (Spain) and Ángel Plaza (Spain), Paolo Perfetti (Italy), Raul A. Simon (Chile), and the proposer.

Answers

Solutions to the Quickies from page 311.

A993. Let $L = \sup\{|f(x_0) - f(y_0)| : f \in A\}$. We prove that $L = d(x_0, y_0)$. For any $f \in A$, we have $|f(x_0) - f(y_0)| \le d(x_0, y_0)$. Thus $L \le d(x_0, y_0)$. To prove the reverse inequality, define $g : X \to R$ by $g(x) = d(x, x_0)$. Then for x and y in X,

$$|g(x) - g(y)| = |d(x, x_0) - d(y, x_0)| \le d(x, y),$$

which proves that $g \in A$. Therefore

$$\begin{aligned}L &= \sup\{|f(x_0) - f(y_0)| : f \in A\} \\ &\ge |g(x_0) - g(y_0)| = |d(x_0, x_0) - d(x_0, y_0)| = d(x_0, y_0).\end{aligned}$$

Thus, $L \ge d(x_0, y_0)$, and it follows that $L = d(x_0, y_0)$.

A994. Let I denote the value of the integral. Note that if z is a real number and z is not an integer, then $\{z\} + \{-z\} = 1$. By symmetry

$$I = \int_0^1\int_0^1 \left\{\left(\frac{1}{x}\right)^k - \left(\frac{1}{y}\right)^k\right\} dx\,dy = \int_0^1\int_0^1 \left\{\left(\frac{1}{y}\right)^k - \left(\frac{1}{x}\right)^k\right\} dx\,dy.$$

Hence,

$$\begin{aligned}I &= \frac{1}{2}(I + I) \\ &= \frac{1}{2}\int_0^1\int_0^1 \left(\left\{\left(\frac{1}{x}\right)^k - \left(\frac{1}{y}\right)^k\right\} + \left\{\left(\frac{1}{y}\right)^k - \left(\frac{1}{x}\right)^k\right\}\right) dx\,dy \\ &= \frac{1}{2}\int_0^1\int_0^1 1\,dx\,dy = \frac{1}{2},\end{aligned}$$

because the set on which the integrand is 0 is a set of measure 0.