Math 145 **Fall 2009**

Practice Test 1 - Solutions

1. (a) $\forall x P(-x, x)$ is false: e.g. for $x = -3$, $-(-3) \not\leq -3$.
   (b) $\exists x \exists y P(x, y)$ is true: e.g. for $x = 3$ and $y = 4$, $3 \leq 4$.
   (c) $\forall x \exists y P(x, y)$ is true: for any x, let $y = x$, then $x \leq y$ holds.
   (d) $\exists x \forall y P(x, y)$ is false: no matter what the value of x is, for $y = x - 1$ we have $x \not\leq y$.
   (e) $\forall x \forall y P(x, y)$ is false: e.g. for $x = 4$ and $y = 3$, $4 \not\leq 3$.
2. We will prove this statement by contrapositive, i.e. we will prove that if either n or m is even, then $nm + 2n + 2m$ is even. Assume that either n or m is even. Since the expression $nm + 2n + 2m$ is symmetric with respect to n and m, WLOG we can assume that n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. Then $nm + 2n + 2m = 2km + 4k + 2m = 2(km + 2n + m)$. Since $km + 2n + m \in \mathbb{Z}$, $nm + 2n + 2m$ is even.
   Note: we could also consider the following two cases: (1) n is even; (2) m is even.
3. This statement is false. E.g. 0 is a rational number and $\sqrt{2}$ is an irrational number; their product is $0 \cdot \sqrt{2}$, so it is rational.
4. Proof by Mathematical Induction.
   Basis step: if $n = 1$, then $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$ is true.
   Inductive step: assume the statement holds for $n = k$ for some natural number k. We will show that it holds for $n = k + 1$. In other words, we assume that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

   holds, and we will prove that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

   holds.
   We have:
   $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (k+1)(k+2) =$

   $(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + k(k+1)) + (k+1)(k+2) =$

   $\dfrac{k(k+1)(k+2)}{3} + (k+1)(k+2) = \dfrac{k(k+1)(k+2) + 3(k+1)(k+2)}{3} =$
   $\dfrac{(k+1)(k+2)(k+3)}{3}$.

5. Proof by Strong Mathematical Induction.
Basis step. If $n = 1$, then the identity says that $F_0^2 + F_1^2 = F_1^2$, or $0^2 + 1^2 = 1^2$ which is true.
Inductive step. Assume that it holds for all $1 \le n \le k$. Namely, we will use that it holds for $n = k$ and $n = k - 1$, i.e.

$$F_{k-1}^2 + F_k^2 = F_{2k-1}$$

and $F_{(k-1)-1}^2 + F_{(k-1)}^2 = F_{2(k-1)-1}$, or equivalently,

$$F_{k-2}^2 + F_{k-1}^2 = F_{2k-3}.$$

We want to prove that it holds for $n = k + 1$, i.e. $F_{(k+1)-1}^2 + F_{k+1}^2 = F_{2(k+1)-1}$, or, equivalently,

$$F_k^2 + F_{k+1}^2 = F_{2k+1}.$$

It may be easier here to work from the right hand side.
$F_{2k+1} = F_{2k} + F_{2k-1} = F_{2k-1} + F_{2k-2} + F_{2k-1} = 2F_{2k-1} + F_{2k-2} = 2F_{2k-1} + F_{2k-1} - F_{2k-3} = 3F_{2k-1} - F_{2k-3} = 3(F_{k-1}^2 + F_k^2) - (F_{k-2}^2 + F_{k-1}^2) = 3F_{k-1}^2 + 3F_k^2 - F_{k-2}^2 - F_{k-1}^2 = 2F_{k-1}^2 + 3F_k^2 - F_{k-2}^2 = 2F_{k-1}^2 + 3F_k^2 - (F_k - F_{k-1})^2 = 2F_{k-1}^2 + 3F_k^2 - F_k^2 + 2F_kF_{k-1} - F_{k-1}^2 = F_{k-1}^2 + 2F_k^2 + 2F_kF_{k-1} = F_{k-1}(F_{k-1} + F_k) + F_k(F_k + F_{k-1}) + F_k^2 = F_{k-1}F_{k+1} + F_kF_{k+1} + F_k^2 = (F_{k-1} + F_k)F_{k+1} + F_k^2 = F_k^2 + F_{k+1}^2$.

6. Since there are 52 whole weeks in a year, Kevin is paid at least 26 times a year. Since there are 12 months, by generalized Dirichlet's box principle, at least one month will contain 3 pay days.

7. Note that for each k, the function f^k is a permutation of the set X and there are $5! = 120$ different permutations of the set X. Consider $f, f^2, \dots, f^{121}$. By Dirichlet's box principle, at least two of these are equal, i.e. $f^a = f^b$ for some $a < b$, $a, b \in \mathbb{N}$. Then f^{b-a} is the identity function, i.e. $f^{b-a}(x) = x$ for all $x \in X$.

8. Let the smallest odd integer be $2k + 1$, where $k \in \mathbb{N}$. Then the four consecutive odd integers are $2k + 1$, $2k + 3$, $2k + 5$, and $2k + 7$. Their sum is $8k + 16$. This number can be written as $2(4k + 8)$, thus it is even. Therefore it cannot end with 7.

9. First notice that the value of y must be even, since $100 - 2x$ is even. Also, $y \ge 1$ (since we only count the positive solutions) and $y \le 32$ (since if $y \ge 34$, then $x = \frac{100-3y}{2} \le \frac{100-3\cdot 34}{2} < 0$). But every even value of y between 1 and 32 does produce a positive integer value of x, so there are 32 positive pairs that are solutions to the given equation.