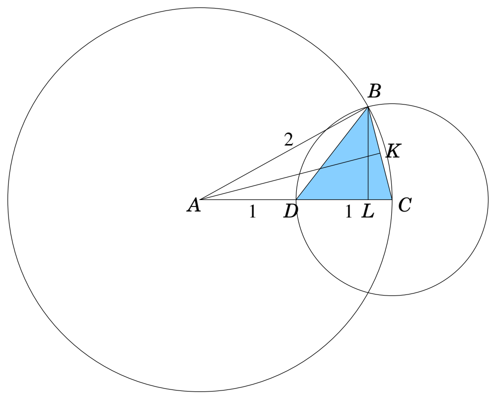
Practice Test 3 - Solutions

- $\log_8 4 = \frac{2}{3}$ *since* $8^{2/3} = 4$

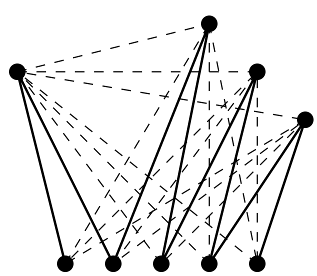
1. A circle of radius 2 passes through the center of a circle of radius 1 (see picture below). Find the area of the shaded triangle.



Label the points as shown in the above picture, where AK and BL are heights of the triangle ABC. Since $AB = AC$, the triangle ABC is isosceles. $|BC| = 1$, therefore $|BK| = \frac{1}{2}$, and by Pythagorean theorem $|AK| = \sqrt{2^2 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$. Using base BC and height AK, the area of triangle ABC is $\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{15}}{2} = \frac{\sqrt{15}}{4}$. On the other hand, using base AC and height BL, its area is $\frac{1}{2} \cdot 2 \cdot |BL| = |BL|$. Therefore $|BL| = \frac{\sqrt{15}}{4}$. Then the area of triangle BCD is $\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{15}}{4} = \frac{\sqrt{15}}{8}$.

2. A graph $K_{k,l,m}$ has $k + l + m$ vertices divided into three sets: k vertices in one set, l vertices in another set, and m vertices in the third set. Two vertices are connected if and only if they are in different sets. Prove that $K_{1,3,5}$ has a Hamilton path but not a Hamilton cycle.

A Hamilton path is shown:



There is no Hamilton cycle because among 9 vertices, 5 are in one set. Thereofe if a Hamilton cycle existed then at least 2 of these 5 would be consecutive in the cycle. However, they cannot be joined because since they are in one set.

3. Find the greatest common divisor d of $a = 96$ and $b = 44$, and integer numbers x and y such that $xa + yb = d$.

   $96 = 2 \cdot 44 + 8$
   $44 = 5 \cdot 8 + 4$
   $8 = 2 \cdot 4$
   Therefore $d = 4$.
   $4 = 44 - 5 \cdot 8 = 44 - 5 \cdot (96 - 2 \cdot 44) = 44 - 5 \cdot 96 + 10 \cdot 44 = 11 \cdot 44 - 5 \cdot 96.$
   So $a = -5$, $b = 11$.

4. Find a number c such that the line $y = c$ divides the region bounded by $y = 5 - x^2$ and the x-axis into two regions of equal area.

   Since the parabola is symmetric about the y*-axis and the line is horizontal, the line also divides the region bounded by* $y = 5 - x^2$*, the* x*-axis, and the* y*-axis into two regions of equal area. Thus we may only consider the first quadrant. Therefore the area of the region between the parabola and the line is* $\frac{1}{2}$ *of the area under the parabola. Let the intersection point of the given parabola and line that lies in the first quadrant be* (a, c)*. Then* $c = 5 - a^2$*, and we have:*
   $$\int_0^a (5 - x^2 - c)dx = \frac{1}{2}\int_0^{\sqrt{5}} (5 - x^2)dx$$
   $$\int_0^a (a^2 - x^2)dx = \frac{1}{2}\int_0^{\sqrt{5}} (5 - x^2)dx$$
   $$2\int_0^a (a^2 - x^2)dx = \int_0^{\sqrt{5}} (5 - x^2)dx$$
   $$2\left(a^2x - \frac{x^3}{3}\right)\Big|_0^a = \left(5x - \frac{x^3}{3}\right)\Big|_0^{\sqrt{5}}$$
   $$2\left(a^3 - \frac{a^3}{3}\right) = 5\sqrt{5} - \frac{5\sqrt{5}}{3}$$
   $$\frac{4a^3}{3} = \frac{10\sqrt{5}}{3}$$
   $$a^3 = \frac{5\sqrt{5}}{2}$$
   $$a = \frac{\sqrt{5}}{\sqrt[3]{2}}$$
   $$c = 5 - a^2 = 5 - \frac{5}{\sqrt[3]{4}}$$

- Find a curve that passes through the point $(3, 2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected by P.

  Let the curve be given by $y = f(x)$*. Since it passes through* $(3, 2)$*,* $f(3) = 2$*.*
  At a point $P(a, f(a))$*, the tangent line has slope* $f'(a)$*, and equation* $y - f(a) = f'(a)(x - a)$*. Its* x*-intercept is* $\left(-\frac{f(a)}{f'(a)} + a, 0\right)$*. The part of the tangent line that lies in the first quadrant is bisected by* P *iff* $2a = -\frac{f(a)}{f'(a)} + a$*. Thus* $af'(a) = -f(a)$*. Since this must be true for every point on the curve in the first quadrant, we have the differential equation* $xf'(x) = -f(x)$*. Any function of the form* $f(x) = \frac{c}{x}$ *is a solution of this equation. Using the condition* $f(3) = 2$*, we find* $c = 6$*. So* $f(x) = \frac{6}{x}$ *satisfies the required condition.*