Math 145 **Fall 2009**

Test 1 - Solutions

1. We will prove this statement by contradiction. Suppose that $\sqrt[3]{5}$ is rational. Then it can be written as a fraction in lowest terms:
$\sqrt[3]{5} = \dfrac{a}{b}$ where $a, b \in \mathbb{Z}$, $(a, b) = 1$.
Then $\sqrt[3]{5}b = a$, so $5b^3 = a^3$.
Therefore $5|a^3$. The prime factorization of a^3 consists of all primes in the prime factorization of a, each repeated three times. Since the prime factorization of a^3 contains 5, the prime factorization of a must contain 5 as well. So $5|a$. Then $a = 5k$ for some $k \in \mathbb{Z}$. It follows that $5b^3 = (5k)^3$, or $b^3 = 5^2k^3$. Thus the prime factorization of b^3 contains 5, and for the same reason as above, the prime factorization of b contains 5. Then $5|b$ and $(a, b) \neq 1$. Contradiction.
Thus our assumption that $\sqrt[3]{5}$ is rational was false.
<u>Note:</u> an alternative argument for "if $5|a^3$, then $5|a$" can be made by contrapositive. Namely, if $5 \not| a$, consider 4 cases:
Case I: $a \equiv 1 \pmod 5$, then $a^3 \equiv 1 \pmod 5$, so $5 \not| a^3$.
Case II: $a \equiv 2 \pmod 5$, then $a^3 \equiv 3 \pmod 5$, so $5 \not| a^3$.
Case III: $a \equiv 3 \pmod 5$, then $a^3 \equiv 2 \pmod 5$, so $5 \not| a^3$.
Case IV: $a \equiv 4 \pmod 5$, then $a^3 \equiv 4 \pmod 5$, so $5 \not| a^3$.
<u>Note:</u> in the above proof with 4 cases, it is OK to use $a = 4q + 1$ instead of $a \equiv 1 \pmod 5$, etc.
<u>Grading:</u> the proof is worth 90%, the answer to the question about the type of proof is 10%.
<u>Typical flaw:</u> fail/forget to explain why $5|a^3$ implies $5|a$. This explanation is considered to be 25% of the proof.
<u>Typical work:</u> first three lines are good, then stop or nonsense: 10%.
first four lines, then stop or nonsens: 20%

2. <u>Note.</u> As you probably noticed, there is a typo in the identity. It should be:
$$F_0 - F_1 + F_2 - F_3 + \ldots - F_{2n-1} + F_{2n} = F_{2n-1} - 1.$$
<u>Proof by Mathematical Induction.</u>
Basis step. If $n = 1$, the identity is $F_0 - F_1 + F_2 = F_1 - 1$ which is true since $0 - 1 + 1 = 1 - 1$.
Inductive step. Assume the identity holds for $n = k$ for some $k \in \mathbb{N}$. We will prove that it holds for $n = k + 1$.
Indeed, if $F_0 - F_1 + F_2 - F_3 + \ldots - F_{2k-1} + F_{2k} = F_{2k-1} - 1$, then
$F_0 - F_1 + F_2 - F_3 + \ldots - F_{2(k+1)-1} + F_{2(k+1)} =$

$F_0 - F_1 + F_2 - F_3 + \ldots - F_{2k-1} + F_{2k} - F_{2k+1} + F_{2k+2} =$
$F_{2k-1} - 1 - F_{2k+1} + F_{2k+2} = F_{2k-1} - 1 + F_{2k} = F_{2k+1} - 1 = F_{2(k+1)-1} - 1.$
Grading: stated that will prove by Mathematical Induction and did the basis step correctly: 30%.
Inductive step is worth 70%.
In the inductive step:
statement "assume holds for $n = k$ for $k \in \mathbb{N}$, will prove that it holds for $n = k+1$": 10%
stated above and refrased correctly: "assume $F_0 - F_1 + F_2 - F_3 + \ldots - F_{2k-1} + F_{2k} = F_{2k-1} - 1$,, will prove $F_0 - F_1 + F_2 - F_3 + \ldots - F_{2(k+1)-1} + F_{2(k+1)} = F_{2(k+1)-1} - 1$, but did not prove: 20

3. Partition the given set into two subsets: $\{2, 4, 8, 16, 32, 64, 128\}$ and $\{5, 25, 125\}$. Since three numbers are chosen, by Dirichlet's box principle, at least two of them will be in the same subset. If they are both in the first subset, then they both are powers of 2; if they are in the second subset, they both are powers of 5. In any case, the larger of these two numbers is divisible by the smaller, say, a_i is divisible by a_j. Then the quotient a_i/a_j is an integer.
   Typical mistake: consider an example or even a few examples and show, for these specific examples, that at least one of the quotients is an integer. This would receive 0%.
   Note: considering all possible pairs from the first subset and all possible pairs from the second subset and showing that for each pair one of the two quotients is an integer (without stating that the numbers in these subsets are powers of 2 and of 5 respectively) is OK, although time/space consuming; conversely, giving the argument about "two types of numbers" (powers of 2 and powers of 5) without explicitly giving both subsets is OK as well, if the argument is clearly written.

4. Observe that $3^4 \equiv 81 \equiv 1 (\text{mod } 10)$, therefore $3^{2009} \equiv (3^4)^{502} \cdot 3 \equiv 1 \cdot 3 \equiv 3 \ (\text{mod } 10)$.
   Also, $2^5 \equiv 32 \equiv 2 (\text{mod } 10)$, therefore $2^{2009} \equiv (2^5)^{401} \cdot 2^4 \equiv 2^401 \cdot 2^4 \equiv 2^{405} \equiv (2^5)^{81} \equiv 2^{81} \equiv (2^5)^{16} \cdot 2 \equiv 2^{16} \cdot 2 \equiv 2^{17} \equiv (2^5)^3 \cdot 2^2 \equiv 2^3 \cdot 2^2 \equiv 2^5 \equiv 2 \ (\text{mod } 10)$.
   Thus $2^{2009} + 3^{2009} \equiv 2 + 3 \equiv 5 \ (\text{mod } 10)$, so the last digit of $2^{2009} + 3^{2009}$ is 5.
   Note: there are many other ways to get both of the above parts (2^{2009} and 3^{2009}). Any correct way is acceptable.
   Grading: stated that need to calculate the given expression modulo 10: 10%; each of the calculations (2^{2009} and 3^{2009}): 40%; added the two parts: 10%.
   Typical alternative approach: calculate the last digits of 2^1, 2^2, 2^3, etc. Get the sequence: $2, 4, 8, 6, 2, 4, 8, 6, \ldots$. Observe that the sequence is periodic with period 4. Since $2009 \equiv 1 (\text{mod } 4)$, the last digit of 2^{2009} is 2. Similarly, for powers of 3 get the sequence $3, 9, 7, 1, 3, 9, 7, 1, \ldots$. So the last digit of 3^{2009} is 3. The rest is as above.

5. Let's denote the terms of these sequence $s_0, s_1, s_2, \ldots$. Consider the first 10001 pairs of consecutive terms: (s_0, s_1), (s_1, s_2), (s_{10000}, s_{10001}). Since each s_i has 100 possibile values (from 0 to 99, since addition is done modulo 100), each pair has $100^2 = 10000$ possible values. Since there are more pairs than possible values, by Dirichlet's box principle there must be repetition. Let k and n be such that $k < n$ and $(s_k, s_{k+1}) = (s_n, s_{n+1})$. Then $s_{k+2} = s_{n+2}$, $s_{k+3} = s_{n+3}$, etc., so the piece $s_k, s_{k+1}, s_{k+2}, \ldots s_{n-1}$ will repeat. Since any term of the sequence is always the sum of the two previous terms, this piece will repeat infinitely.

   Note: this is a tricky problem on undertanding of all of the following: how the Fibonacci sequence works, how addition modulo n works, and Dirichlet's box principle (need to identify what to use as "objects" and figure out how many "boxes" there are and thus how many "objects" are needed to apply the principle). A rigorous proof such as by math induction of the periodicity is not expected, but is welcome. However, the appropriate "boxes" and "objects" and the correct number of these are required to get full credit. Partial credit is given for some good ideas.