MATH 145
Test 3 - Solutions

1. Consider the standard chessboard coloring. The chessboard has 32 white squares and 32 black squares. Since two of the four corner squares are white and two are black, after they are removed, 30 white and 30 black squares remain. Assume that it is possible to cover the obtained board by T-tetrominoes. Since there are 60 squares total and each tetromino covers 4, 15 tetrominoes must be used. Notice that no matter how a T-tetromino is placed on a board with chessboard coloring, it covers either 1 white and 3 black squares or 3 white and 1 black squares. Let n be the number of tetrominoes covering 1 white and 3 black squares each. Then $15 - n$ tetrominoes cover 3 white and 1 black squares each. Now, counting the total number of white squares covered, we have:
   $n + 3(15 - n) = 30$
   $n + 45 - 3n = 30$
   $15 = 2n$.
   This equation has no integer solutions. We get a contradiction, thus such a covering is not possible.

   Typical mistake: assuming that since each T-tetromino covers either 1 white and 3 black squares or 3 white and 1 black squares, it follows that 15 T-tetrominoes cover either 15 white and 45 black squares or 45 white and 15 black squares. This is not necessarily the case.

   Typical flaw: using the fact that 15 times an odd number is odd (this, in fact, is equivalent to the previous mistake). This is insufficient. What we really need here is that the sum of 15 odd numbers is odd, which is a stronger statement than the one about the product.

2. Since $3|12$, the given condition implies $a_n \equiv a_{n-1} + a_{n-2} + a_{n-3} \pmod 3$. Then $a_{n-3} \equiv a_n - a_{n-1} + a_{n-2} \pmod 3$. Suppose that the triple $3, 6, 9$ does occur. Since all of the numbers 3, 6, and 9 are congruent to 0 modulo 3, the number that appears before 3 is also congruent to 0 modulo 3. And the number before it. And the number before that. We see that all of the terms in the sequence must then be congruent to 0 modulo 3. (Formally, this statement can be proved by induction.) However, the terms 1, 2, 11, and 8 are not. Contradiction.

   Typical mistake: working out a number of terms of the given sequence and then just stating without proof that the triple 3, 6, 9 does not occur. Or saying something like that the numbers in the sequence have all kinds of remainders upon division by 3, and three consecutive terms are never all divisible by 3 - again, need to prove that.

3. Consider the following graphs: it has 6 vertices representing 6 children, and two vertices are connected by an edge if and only if these two siblings take a class together this semester. Then, for each child, the number of siblings that they take a class with, is the degree of the corresponding vertex. By a theorem discussed in class, the sum of the

degrees of all vertices must be even. Since $1+1+1+2+3+3=11$ is odd, these numbers cannot represent degrees of all vertices of a graph.

Typical mistake: trying to draw such a graph and stating that it is impossible because one is not able to draw it.

4. Let (a,b) be the point where the line is tangent to the circle. First, the point lies on the circle, so $a^2+b^2=1$. Second, the slope of the line is equal to the slope of the circle at this point. The slope of the line is $\frac{2-b}{2-a}$. There are multiple ways to find the slope of the circle. One is to solve the equation $x^2+y^2=1$ for y and differentiate the obtained function. Another is to use implicit differentiation. Yet another is to use the fact that the product of the slopes of perpendicular lines is equal to -1. Using the last method, since the slope of the radius to the point (a,b) is $\frac{b}{a}$, the slope of the tangent line is $-\frac{a}{b}$. So we have
   $\frac{2-b}{2-a}=-\frac{a}{b}$
   $(2-b)b=-a(2-a)$
   $2b-b^2=-2a+a^2$
   $2b+2a=a^2+b^2$
   $2a+2b=1$
   $a+b=\frac{1}{2}$
   $b=\frac{1}{2}-a$
   Now substituting this into $a^2+b^2=1$, we get
   $a^2+\left(\frac{1}{2}-a\right)^2=1$
   $a^2+\frac{1}{4}-a+a^2-1=0$
   $2a^2-a-\frac{3}{4}=0$
   $8a^2-4a-3=0$
   $a=\frac{4\pm\sqrt{16+96}}{16}$
   $a=\frac{4\pm\sqrt{112}}{16}$
   $a=\frac{4\pm4\sqrt{7}}{16}$
   $a=\frac{1\pm\sqrt{7}}{4}$.
   If $a=\frac{1+\sqrt{7}}{4}$, $b=\frac{1}{2}-\frac{1+\sqrt{7}}{4}=\frac{1-\sqrt{7}}{4}$;
   if $a=\frac{1-\sqrt{7}}{4}$, $b=\frac{1}{2}-\frac{1-\sqrt{7}}{4}=\frac{1+\sqrt{7}}{4}$.
   The slopes are then $-\frac{a}{b}=-\frac{1+\sqrt{7}}{1-\sqrt{7}}=\frac{\sqrt{7}+1}{\sqrt{7}-1}$ and $-\frac{1-\sqrt{7}}{1+\sqrt{7}}=\frac{\sqrt{7}-1}{\sqrt{7}+1}$ respectively.

- **For extra credit:** Hint: stretch both ellipses by a factor of 2 vertically. They become circles. The area of their intersection is multiplied by 2. We have found the area of the intersection of two circles in class.