PROBLEMS

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PROPOSALS

To be considered for publication, solutions should be received by November 1, 2013.

1921. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia and Elton Bojaxhiu, Kriftel, Germany.*

Let $f : (0, \infty) \to \mathbb{R}$ be a function such that

$$\frac{1}{2}\left(f(\sqrt{x}) + f(\sqrt{y})\right) = f\left(\sqrt{\frac{x+y}{2}}\right)$$

for every $x, y \in (0, \infty)$. Prove that

$$\frac{1}{n}\left(f(\sqrt{x_1}) + f(\sqrt{x_2}) + \cdots + f(\sqrt{x_n})\right) = f\left(\sqrt{\frac{x_1 + x_2 + \cdots + x_n}{n}}\right)$$

for every positive integer n and for every $x_1, x_2, \ldots, x_n \in (0, \infty)$.

1922. *Proposed by Arkady Alt, San Jose, CA.*

Let m_a, m_b, and m_c be the lengths of the medians of a triangle with circumradius R and inradius r. Prove that

$$m_a m_b + m_b m_c + m_c m_a \le 5R^2 + 2Rr + 3r^2.$$

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We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1923. *Proposed by Leonid Menikhes and Valery Karachik, South Ural State University, Chelyabinsk, Russia.*

Let m and n be nonnegative integers. Find a closed-form expression for the sum

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{2m}{m-n+k}.$$

1924. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Find a necessary and suffcient condition on (a_1, a_2, a_3, a_4) for the series

$$\sum_{n=0}^{\infty}\left(\frac{a_1}{4n+1}+\frac{a_2}{4n+2}+\frac{a_3}{4n+3}+\frac{a_4}{4n+4}\right)$$

to converge, and determine the sum of this series when that condition is satisfied.

1925. *Proposed by Tim Kröger and Rudolf Rupp, Georg Simon Ohm University of Applied Sciences, Nürnberg, Germany.*

Probably every mathematician teaching undergraduate mathematics has experienced the difficulty of persuading every student that the equation $(A+B)^{-1} = A^{-1} + B^{-1}$ is not true for arbitrary matrices A and B. However, the equation is true for *some* matrices A and B.

For every positive integer n, determine all pairs of $n \times n$ real matrices A and B such that $(A+B)^{-1} = A^{-1} + B^{-1}$.

Quickies

Answers to the Quickies are on page 233.

Q1031. *Proposed by Herman Roelants, Institute of Philosophy, University of Leuven, Belgium.*

For which positive integers n do there exist positive integer solutions x, y to the diophantine equation $4xy - x + y = n$?

Q1032. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Suppose that f is a continuous real-valued function on $[a, b]$ and $c \in (a, b)$. In addition, suppose that f' exists and is decreasing on (a, b). Prove that

$$(b-c)f(a) + (c-a)f(b) \le (b-a)f(c).$$

Solutions

A Fresnelian definite integral **June 2012**

1896. *Proposed by Timothy Hall, PQI Consulting, Cambridge, MA.*

Find with proof the value of

$$\int_0^{\infty}\frac{\cos(\sqrt{x})}{\sqrt{x}}\cos x\,dx.$$

I. *Solution by Missouri State University Problem Solving Group, Missouri State University, Springfield, MO.*
Substituting $u = \sqrt{x}$ and noting that the integrand is an even function gives

$$\int_0^\infty \frac{\cos(\sqrt{x})}{\sqrt{x}} \cos x \, dx = 2\int_0^\infty \cos(u)\cos(u^2)du = \int_{-\infty}^\infty \cos(u)\cos(u^2)du.$$

Using the addition and subtraction formulas for cosine, completing the square, and using the substitutions $v = u + 1/2$ and $w = u - 1/2$, gives

$$\begin{aligned}
\int_{-\infty}^\infty \cos(u)\cos(u^2)du &= \frac{1}{2}\int_{-\infty}^\infty \left[\cos(u^2+u) + \cos(u^2-u)\right] du \\
&= \frac{1}{2}\int_{-\infty}^\infty \left[\cos\left(\left(u+\frac{1}{2}\right)^2 - \frac{1}{4}\right) + \cos\left(\left(u-\frac{1}{2}\right)^2 - \frac{1}{4}\right)\right] du \\
&= \frac{1}{2}\int_{-\infty}^\infty \cos\left(v^2 - \frac{1}{4}\right)dv + \frac{1}{2}\int_{-\infty}^\infty \cos\left(w^2 - \frac{1}{4}\right)dw \\
&= \int_{-\infty}^\infty \cos\left(v^2 - \frac{1}{4}\right)dv \\
&= \int_{-\infty}^\infty \left[\cos(v^2)\cos\left(\frac{1}{4}\right) + \sin(z^2)\sin\left(\frac{1}{4}\right)\right]dv \\
&= \cos\left(\frac{1}{4}\right)\int_{-\infty}^\infty \cos(v^2)dv + \sin\left(\frac{1}{4}\right)\int_{-\infty}^\infty \sin(v^2)dv.
\end{aligned}$$

Since the Fresnel integrals

$$\int_{-\infty}^\infty \cos(v^2)dv = \sqrt{\frac{\pi}{2}} = \int_{-\infty}^\infty \sin(v^2)dv$$

are well known, the integral in question equals $(\cos(1/4) + \sin(1/4))\sqrt{\pi/2}$.

II. *Solution by Khristo N. Boyadzhiev, Ohio Northern University, Ada, OH.*
As in the first solution, the requested integral is equal to $\int_{-\infty}^\infty \cos(u)\cos(u^2)du$. Consider the real-valued function

$$y(t) = \int_{-\infty}^\infty e^{-ax^2}\cos(xt)dx,$$

where a is a complex constant with $\Re(a) > 0$. Because the integral is absolutely convergent, it is possible to differentiate inside the integral to get

$$y'(t) = -\int_{-\infty}^\infty x e^{-ax^2}\sin(xt)dx.$$

Integrating $y(t)$ by parts gives

$$\begin{aligned}
y(t) &= \int_{-\infty}^\infty e^{-ax^2}\frac{d}{dx}\left(\frac{\sin(xt)}{t}\right)dx \\
&= \lim_{R\to\infty} \frac{1}{t}e^{-ax^2}\sin(xt)\Big|_{x=-R}^{x=R} + \int_{-\infty}^\infty \frac{2ax}{t}e^{-ax^2}\sin(xt)dx \\
&= 0 + \frac{2a}{t}\int_{-\infty}^\infty xe^{-ax^2}\sin(xt)dx = -\frac{2a}{t}y'(t).
\end{aligned}$$

The separable differential equation $dy/dt = (-t/2a)y$ in the variable t has the solution $y(t) = Me^{-t^2/4a}$. To evaluate the constant M, we set $t = 0$ and use the well-known value of the Gaussian integral,

$$y(0) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.$$

Thus

$$\int_{-\infty}^{\infty} e^{-ax^2} \cos(xt) dx = \sqrt{\frac{\pi}{a}}\, e^{-t^2/4a}$$

holds for all $t \in \mathbb{R}$ and for all complex numbers a with $\Re(a) > 0$. By continuity, the equation also holds for $\Re(a) = 0$ as long as $a \neq 0$. Setting $t = 1$ and $a = i$ gives

$$\begin{aligned}\int_{-\infty}^{\infty} (\cos(x^2) - i \sin(x^2)) \cos(x) dx &= \int_{-\infty}^{\infty} e^{-ix^2} \cos(x) dx = \sqrt{\frac{\pi}{i}}\, e^{-1/4i} \\ &= \sqrt{\pi}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\left(\cos\left(\frac{1}{4}\right) + i \sin\left(\frac{1}{4}\right)\right).\end{aligned}$$

Comparing real parts gives

$$\int_{-\infty}^{\infty} \cos(x) \cos(x^2) dx = \sqrt{\frac{\pi}{2}}\left(\cos\left(\frac{1}{4}\right) + \sin\left(\frac{1}{4}\right)\right).$$

Also solved by M. Reza Akhlaghi; George Apostolopoulos (Greece); William C. Bauldry; M. Benito, Ó. Ciaurri, E. Fernández, and L. Roncal; Gerald E. Bilodeau; Robert Calcaterra; Hongwei Chen; Paul Deiermann; Eugene S. Eyeson; Fisher Problem Group; John N. Fitch; Ovidiu Furdui (Romania); J. A. Grzesik; Eugene A. Herman; Julio C. Herrera and Mariano Perez; Omran Kouba (Syria); Isaac Edward Leonard (Canada); Ryan Q. McCluskey; Matthew McMullen; Rituraj Nandan; José Heber Nieto (Venezuela); Northwestern University Math Problem Solving Group; Moubinool Omarjee (France); Tomas Persson and Mikael P. Sundqvist (Sweden); José M. Pacheco (Spain) and Ángel Plaza (Spain); Paolo Perfetti (Italy); Mohammad Riazi-Kermani; Kendall Richards and Therese Shelton; Nicholas C. Singer; Thomas Steinberger; Nora Thornber; Tiberiu Trif (Romania); Michael Vowe (Switzerland); Stan Wagon; Haohao Wang and Jerzy Woydylo; A. David Wunsch; Yanping Xia; Li Zhou; and the proposer.

Partitions with balanced sums **June 2012**

1897. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let n and m be positive integers such that $m < n$. Determine necessary and sufficient conditions for a sequence $\{x_j\}_{j=1}^n$ of real numbers to satisfy that

$$\left|\sum_{j\in S} x_j\right| = \left|\sum_{\substack{1\le j\le n\\ j\notin S}} x_j\right|,$$

for every m-element subset S of $\{1, 2, \dots, n\}$.

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA.
When $m \neq n/2$, the condition is $\sum_{j=1}^n x_j = 0$; when $m = n/2$, the condition is $\sum_{j=1}^n x_j = 0$ or $x_1 = x_2 = \cdots = x_n$. The sufficiency of these conditions can be readily confirmed. Now suppose $\{x_j\}_{j=1}^n$ is a sequence of real numbers satisfying the condition given in the problem. If for some m-element subset S of $\{1, 2, \dots, n\}$,

$$\sum_{j\in S} x_j = -\sum_{\substack{1\le j\le n\\ j\notin S}} x_j,$$

then $\sum_{j=1}^{n} x_j = 0$. Otherwise,

$$\sum_{j \in S} x_j = \sum_{\substack{1 \le j \le n \\ j \notin S}} x_j$$

for every m-element subset S of $\{1, 2, \ldots, n\}$. If for some fixed S_1, there exist $i \in S_1$ and $j \notin S_1$ such that $x_i \neq x_j$, then for $S_2 = (S_1 \setminus \{i\}) \cup \{j\}$,

$$\sum_{j \in S_2} x_j \neq \sum_{\substack{1 \le j \le n \\ j \notin S_2}} x_j .$$

Therefore, every element in S_1 equals every element in $\{1, 2, \ldots, n\} \setminus S_1$, and so $x_1 = x_2 = \cdots = x_n$ and $m = n/2$.

Also solved by George Apostolopoulos (Greece), Jeffrey Boerner and Natacha Fontes-Merz, Paul Budney, Bruce S. Burdick, Robert Calcaterra, Con Amore Problem Group (Denmark), Dmitry Fleischman, Michael Goldenberg and Mark Kaplan, Omran Kouba (Syria), Missouri State University Problem Solving Group, Jaeik Oh (Korea), Texas State University Problem Solvers Group, and the proposer. There were three incorrect solutions.

More on Kuratowski 14-sets — June 2012

1898. *Proposed by Mark Bowron, Laughlin, NV.*

A subset E of a topological space X is called a *Kuratowski* 14-*set* if 14 distinct sets can be obtained by repeatedly applying closure and complement to E in some order. It is known that Kuratowski 14-sets E with $|E| = 3$ exist. Do any exist with $|E| < 3$?

Solution by Bruce S. Burdick, Roger Williams University, Bristol, RI.
The answer is no. Suppose we had such an E. Using c for closure and i for interior, the sets E, cE, icE, $cicE$, iE, ciE, and $iciE$ must all be distinct. (The other seven sets are the complements of these. Note that $iE = X \setminus c(X \setminus E)$.)

The set iE must not be E and it must not be empty, since that would imply that $ciE = iE$. So that rules out $|E| < 2$. Assume then that $E = \{x, y\}$ and $iE = \{x\}$. Note that if $icE \subseteq ciE$, then it would follow that $icE = iciE$. So there is some $z \in icE$ with $z \notin ciE = c\{x\}$. But $z \in cE$, so it must be that $z \in c\{y\}$. Therefore, $y \in icE$. Since x is isolated, we have $E \subseteq icE$, hence $cE \subseteq cicE$. Therefore, $cE = cicE$, a contradiction.

Also solved by Alex Aguado, George Apostolopoulos (Greece), Jeffrey Boerner, Robert Calcaterra, José H. Nieto (Venezuela), and the proposer. There was one incomplete submission.

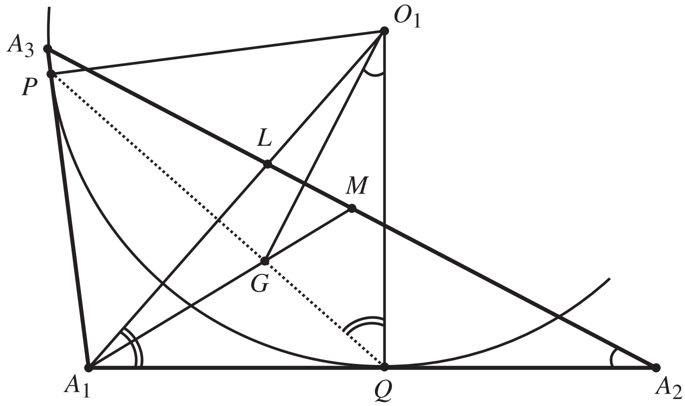
Tangent points collinear with the centroid — June 2012

1899. *Proposed by Michel Bataille, Rouen, France.*

Let $A_1A_2A_3$ be a triangle with centroid G. For $i \in \{1, 2, 3\}$, the circle $\mathcal{C}_i$ with center O_i and radius r_i is tangent to the two lines through A_i spanned by the sides of the triangle; moreover, the points of tangency and G are collinear. Prove that

$$\frac{r_1}{r_1 + GO_1} + \frac{r_2}{r_2 + GO_2} + \frac{r_3}{r_3 + GO_3} = 2.$$

Solution by Shohruh Ibragimov (student), Lyceum Nr2 under the SamIES, Samarkand, Uzbekistan.



Denote by M the midpoint of $\overline{A_2A_3}$, and by P and Q the tangent points of the circle $\mathcal{C}_1$ with the lines A_1A_3 and A_1A_2, respectively. Denote by L the intersection of the bisector of the angle $\angle A_2A_1A_3$ and $\overline{A_2A_3}$. By the Generalized Angle Bisector Theorem applied to $\triangle A_1A_2A_3$ and M, and the Law of Sines applied to $\triangle A_1A_2A_3$, it follows that

$$1 = \frac{A_3M}{MA_2} = \frac{A_1A_3 \cdot \sin\angle MA_1A_3}{A_1A_2 \cdot \sin\angle A_2A_1M} = \frac{\sin\angle A_2}{\sin\angle A_3} \cdot \frac{\sin\angle MA_1A_3}{\sin\angle A_2A_1M}.$$

Again, the Generalized Angle Bisector Theorem applied to isosceles triangles QPA_1 and PQO_1 and point G implies that

$$\frac{\sin\angle MA_1A_3}{\sin\angle A_2A_1M} = \frac{\sin\angle GA_1P}{\sin\angle QA_1G} = \frac{PG}{GQ} = \frac{\sin\angle PO_1G}{\sin\angle GO_1Q}.$$

The last two equations imply that

$$\frac{\sin(\angle A_1 + \angle A_2)}{\sin\angle A_2} = \frac{\sin\angle A_3}{\sin\angle A_2} = \frac{\sin\angle PO_1G}{\sin\angle GO_1Q}.$$

Because $\angle QPO_1 = \angle O_1QP = \frac{1}{2}\angle A_1$, it follows that $\angle PO_1G = \pi - (\angle A_1 + \angle GO_1Q)$, and so

$$\frac{\sin(\angle A_1 + \angle A_2)}{\sin\angle A_2} = \frac{\sin(\angle A_1 + \angle GO_1Q)}{\sin\angle GO_1Q}.$$

Thus $\angle GO_1Q = \angle A_2$. In addition, $\angle O_1QG = \angle O_1QP = \frac{1}{2}\angle A_1 = \angle A_2A_1L$, therefore triangles A_1LA_2 and QGO_1 are similar. Hence

$$\frac{LA_2}{A_1A_2} = \frac{GO_1}{QO_1} = \frac{GO_1}{r_1}.$$

Once more, by the Angle Bisector Theorem applied to $\triangle A_1A_2A_3$, it follows that $(A_2A_3 - LA_2)/LA_2 = A_3L/LA_2 = A_1A_3/A_1A_2$, and so

$$LA_2 = \frac{A_1A_2 \cdot A_2A_3}{A_1A_2 + A_1A_3}.$$

Using this we obtain $GO_1/r_1 = A_2A_3/(A_1A_2 + A_1A_3)$, and thus

$$\frac{r_1}{r_1 + GO_1} = \frac{1}{1 + GO_1/r_1} = \frac{A_1A_2 + A_1A_3}{A_1A_2 + A_2A_3 + A_1A_3}.$$

Adding similar identities for the expressions $r_2/(r_2 + GO_2)$ and $r_3/(r_3 + GO_3)$, we get

$$\sum_{i=1}^{3} \frac{r_i}{r_i + GO_i} = \frac{\sum_{\text{cyc}}(A_1A_2 + A_1A_3)}{A_1A_2 + A_2A_3 + A_1A_3} = 2.$$

Also solved by George Apostolopoulos (Greece), Robert Calcaterra, Chip Curtis, Michael Goldenberg and Mark Kaplan, L. R. King, Omran Kouba (Syria), Kee-Wai Lau (China), Peter Nüesch (Switzerland), Traian Viteam (Uruguay), Michael Vowe (Switzerland), and the proposer. There was one incorrect submission.

Rings that are never isomorphic **June 2012**

1900. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let X be a set, and let S_X denote the set of all functions $f : X \to \mathbb{Z}$. The set S_X becomes a ring via the operations $(f + g)(x) := f(x) + g(x)$ and $(f \cdot g)(x) := f(x)g(x)$. Let B_X be the subring of S_X consisting of the functions f whose images in $\mathbb{Z}$ are finite. Does there exist an infinite set X such that the rings B_X and S_X are isomorphic?

Solution by Paul Budney, Sunderland, MA.

The answer is no. Suppose X is infinite and $\phi : B_X \to S_X$ is a ring isomorphism. Let $h \in S_X \setminus B_X$ be a function whose range is the set of positive integers, and let $f = \phi^{-1}(h)$. If $f \cdot g = \mathbf{0}$ for some nonzero function $g \in S_X$, then $h \cdot \phi(g) = \phi(f) \cdot \phi(g) = \phi(f \cdot g) = \phi(\mathbf{0}) = \mathbf{0}$, which is a contradiction since $\phi(g)$ is not the zero function and $h(x)$ is never zero. It follows that $f(x)$ is never zero, otherwise if $f(x_0) = 0$, then g defined as $g(x_0) = 1$ and $g(x) = 0$ for $x \neq x_0$ verifies that $f \cdot g = \mathbf{0}$. Let $m \neq 0$ be the product of all the numbers in the range of f. Define $j \in B_X$ by $j(x) = m/f(x)$. Then $f \cdot j = \mathbf{m}$, the constant function $\mathbf{m}$. Note that the constant function $\mathbf{1}$ is the multiplicative identity, and thus $\phi(\mathbf{1}) = \mathbf{1}$ and $\phi(\mathbf{m}) = m\phi(\mathbf{1}) = m\mathbf{1} = \mathbf{m}$. It follows that $h \cdot \phi(j) = \phi(f) \cdot \phi(j) = \phi(f \cdot j) = \phi(\mathbf{m}) = \mathbf{m}$. But then every positive integer divides m, which is impossible. Thus B_X and S_X are not isomorphic for any infinite set X.

Also solved by George Apostolopoulos (Greece), Robert Calcaterra, Bruce S. Burdick, Eugene A. Herman, Reiner Martin (Germany), Peter McPolin (Northern Ireland), Texas State University Problem Solvers Group, and the proposer.

Answers

Solutions to the Quickies from page 228.

A1031. The answer is all positive integers n such that $4n - 1$ is a composite integer. Multiplying both sides of $4xy - x + y = n$ by 4 and adding -1 to both sides leads to $(4x + 1)(4y - 1) = 4n - 1$. This implies that $4n - 1$ is a positive odd integer. Conversely, any composite integer of the form $4n - 1$ is always the product in at least one way of an integer of the form $4x + 1$ and an integer of the form $4y - 1$ with $x, y > 0$.

A1032. By the Mean Value Theorem, there exist $d \in (a, c)$ and $e \in (c, b)$ such that

$$\frac{f(c) - f(a)}{c - a} = f'(d) \qquad \text{and} \qquad \frac{f(b) - f(c)}{b - c} = f'(e).$$

Because $d < e$ and f' is decreasing, it follows that

$$\frac{f(b) - f(c)}{b - c} \leq \frac{f(c) - f(a)}{c - a}.$$

Multiplying both sides by $(b - c)(c - a)$ and rearranging gives

$$(b - c)f(a) + (c - a)f(b) \leq (b - a)f(c).$$