The Greatest Two-Point Answer Ever

William Dunham and Hank Sneddon

William Dunham

I have been teaching college mathematics for nearly 40 years, and in that time I've given (and, alas, graded) thousands of exams. It is no small achievement for one particular answer on one particular exam to stand out over all these decades. But one does, and this is its story.

In the spring semester of 2013, I was a visiting professor of mathematics at Harvard, where I had been invited to teach a course on the work of Leonhard Euler. The course title, "Much Ado About Everything," pretty much says it all. Euler stands as a mathematical dynamo whose contributions were, and forever will be, fundamental throughout our discipline. It is next to impossible to take a mathematics class without encountering concepts such as the Euler phi function, the Euler line, the Euler polyhedral formula, or the Euler identity. He pops up in analysis, number theory, and discrete mathematics. He is a fixture in courses both pure and applied. We now live in an age of specialists, but Euler's specialty seemed to be omniscience.

I had designed my class for mathematically inquisitive students in their first or second year of college. I imagined that my audience would display substantial mathematical talent—it was *Harvard,* after all—but would also enjoy something a little different. Thus, the course had historical and even biographical components, but it was primarily an examination of some of Euler's greatest mathematical hits.

Leonhard Euler.

Partway through the semester, when considering

Euler's number theory, I introduced the Euler sigma function. If n is a whole number, we define $\sigma(n)$ to be the sum of all the whole number divisors of n. As an example, $\sigma(12) = 1+2+3+4+6+12 = 28$. I also stressed this function's critical multiplicative property: If a and b are relatively prime, then $\sigma(ab) = \sigma(a)\sigma(b)$.

As an aside, I should mention that Euler actually wrote $\int n$ rather than $\sigma(n)$. Of course this borders on sacrilege, given the exalted status of the integral sign. It would be like using 𝄞 for something other than the G-clef or £ for something besides pounds sterling. Modern mathematicians have abandoned Euler's notation and made the lowercase sigma their symbol of choice for the sum of a whole number's divisors.

Be that as it may, at semester's end I had to prepare a final exam. As all professors know, this involves balancing easier and harder problems that properly reflect the class material. After several false starts and adjustments, I still found myself two points short and so needed to write one more (very easy) question. Thus, I inserted the following:

Challenge. Prove, or disprove via counterexample, the assertion: If n is a whole number, then $\sigma(\sigma(n)) \le 4n$.

I can report that, of my 23 students, all but two instantly disproved this with the counterexample of $n = 6$, for clearly $\sigma(\sigma(6)) = \sigma(12) = 28 > 4 \cdot 6$. One student, who apparently had forgotten the definition of σ, said that the conjecture was true and offered a flawed proof. And one student gave the most impressive answer I had seen in four decades of exams.

The student was Hank Sneddon, a first-year math concentrator from Los Angeles. Hank disproved the conjecture with a curious theorem that he concocted and demonstrated on the spot, namely: If n is an even perfect number, then $\sigma(\sigma(n)) > 4n$.

Our class had studied perfect numbers—i.e., numbers that are the sum of their proper divisors, like $6 = 1+2+3$. In the last proposition of Book IX of the *Elements*, Euclid demonstrated that, if $2^k - 1$ is prime, then the number $n = 2^{k-1}(2^k - 1)$ is perfect. Of course, in this case n is also even. So, Euclid had given the world a sufficient condition for a number to be even and perfect.

After the passage of 20 centuries, Euler came along and showed that Euclid's condition was also necessary. That is, Euler proved that, if n is an even perfect number, then n *must* have the form $n = 2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime. As I had showed in class, Euler's strategy of proof was to employ the sigma function, noting that a number n is perfect precisely when $\sigma(n) = 2n$. (For details, see W. Dunham, *Euler: The Master of Us All*, MAA, 1999, chapter 1.)

So, Hank had encountered perfect numbers before. But he had not seen anything like the inequality that I'd posed as my two-point challenge. And, in spite of the fact that it was meant to be an easy exercise, he furnished a nice proof of his little theorem.

I'll now pass the baton to Hank, who can describe his proof and a few subsequent developments.

Hank Sneddon

Upon seeing this problem on the final, I figured that rather than just plugging in a numerical value, I'd like to understand what $\sigma(\sigma(n))$ actually looked like. At the very least, I wanted to know if it was simple to calculate, and, because I had plenty of time left on the exam, I wasn't too worried that these musings would hurt me. I began to play around and push through the inequalities that led to an answer. Here was my line of attack.

Theorem. If n is an even perfect number, then $\sigma(\sigma(n)) > 4n$.

Assume n is even and perfect. Then we know that $n = 2^{k-1}(2^k - 1)$ where the right-most factor is prime and, further, that $\sigma(n) = 2n$. Thus,

$$\sigma(\sigma(n)) = \sigma(2n) = \sigma(2^k(2^k - 1)) = \sigma(2^k)\sigma(2^k - 1)$$

because 2^k and $2^k - 1$ are relatively prime. We had seen in class that

$$\sigma(2^k) = 1 + 2 + 4 + \cdots + 2^k = 2^{k+1} - 1,$$

and the primality of $2^k - 1$ guarantees that $\sigma(2^k - 1) = 2^k$. Thus, I knew that $\sigma(\sigma(n)) = (2^{k+1} - 1)2^k$. It remained to relate this to the inequality posed by the exam question. To do this, I reasoned that

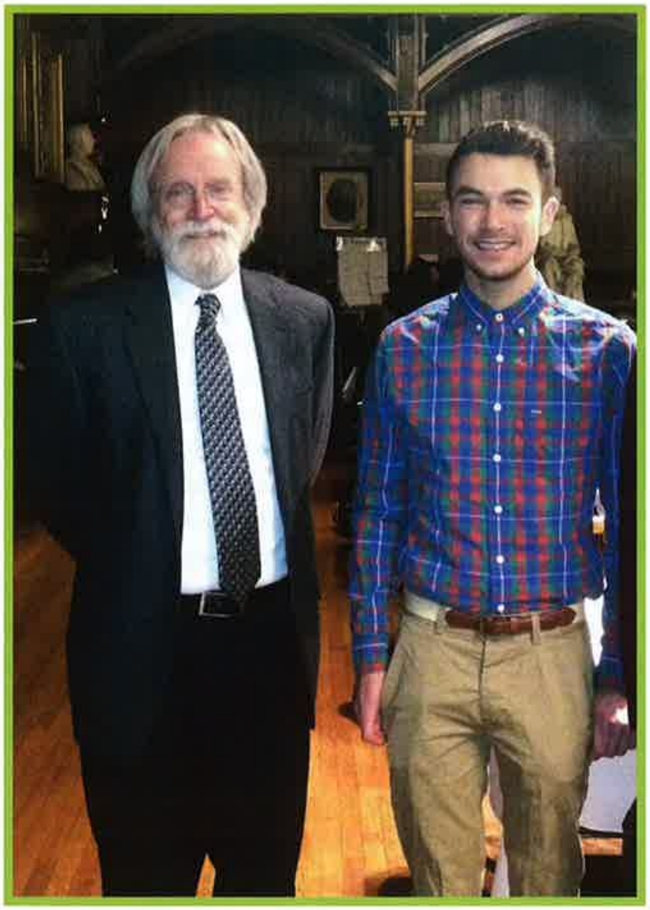
$$2^{k+1} - 1 > 2^{k+1} - 2 = 2(2^k - 1).$$

Consequently,

$$\begin{aligned}\sigma(\sigma(n)) &= (2^{k+1} - 1)2^k \\ &> 2(2^k - 1)2^k \\ &= 4(2^{k-1}(2^k - 1)) \\ &= 4n.\end{aligned}$$

This proved my theorem.

For good measure, I provided a specific numerical example by using the first even perfect number that



William Dunham (left) and Hank Sneddon.

came to mind: $n = 6$. I figured I had earned my two points.

Afterwards, I thought more about this problem and realized that I could find a closed-form expression for $\sigma(\sigma(n))$ that would yield the inequality immediately. My new and improved proposition was:

Theorem. If n is an even perfect number, then

$$\sigma(\sigma(n)) = 4n + \frac{1+\sqrt{1+8n}}{2}.$$

Before proving this theorem, I should concede that it looks strange, for $\sigma(\sigma(n))$ must be a whole number, yet the square root seems to inject a note of irrationality. But n isn't just *any* whole number. Because it is even and perfect, it has a specific form that makes this irrationality an optical illusion.

Starting with Euler's criterion that $n = 2^{k-1}(2^k - 1)$, I first claim that

$$2^k = \frac{1+\sqrt{1+8n}}{2}.$$

This follows because

$$n = 2^{k-1}(2^k - 1) = 2^{2k-1} - 2^{k-1} = \frac{1}{2}((2^k)^2 - 2^k).$$

Thus $(2^k)^2 - 2^k = 2n$, and so $(2^k)^2 - (2^k) - 2n = 0$. The quadratic formula gives

$$2^k = \frac{1+\sqrt{1+8n}}{2},$$

where we use only the positive root because $2^k > 0$. Using our previous calculations and this result, we see that

$$\begin{aligned}
\sigma(\sigma(n)) &= (2^{k+1} - 1)2^k \\
&= 2^{2k+1} - 2^k \\
&= 4(2^{2k-1} - 2^{k-2}) \\
&= 4(2^{k-1}(2^k - 1) + (2^{k-1} - 2^{k-2})) \\
&= 4(n + 2^{k-2}) \\
&= 4n + 2^k \\
&= 4n + \frac{1+\sqrt{1+8n}}{2}.
\end{aligned}$$

And that concludes the proof.

From this theorem, it is obvious that $\sigma(\sigma(n)) > 4n$.

Anyone wishing to push things further will find that, if n is even and perfect, then

$$\sigma(\sigma(\sigma(n))) = \sqrt{1+8n} \cdot \sigma(\sqrt{1+8n}).$$

But I think I'll stop there. After all, this was only a two-point problem. ■

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http://dx.doi.org/10.4169/mathhorizons.22.1.21