THE PLAYGROUND!

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Problem 310. Our first sandbox problem, **Teething Ring,** concerns two concentric circles and, between them, $n > 2$ mutually tangent circles (see figure 1). The concentric circles have radii 1 and $r_n > 1$.

a) Find r_n in terms of n.

b) Let A_n be the area inside the n small circles, and let B_n be the area between the two concentric circles. Then the ratio A_n / B_n gives the *packing density* of the circles trapped between the concentric circles. Find the limit of this ratio as $n \to \infty$.

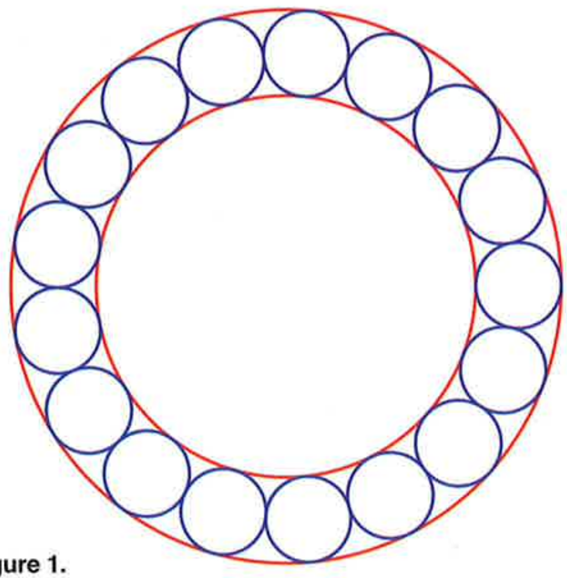


Figure 1.

Problem 311. Our second sandbox problem comes from D. M. Bătinețu-Giurgiu and Neculai Stanciu of Romania. In **Square Sum,** the proposers ask you to find all pairs of positive integers (m,n) satisfying $2^{2016} + 2^{2012} + 2^{2008} + 2^m = n^2$.

THE ZIP-LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Problem 312. In their article "The Greatest Two-Point Answer Ever," William Dunham and Hank Sneddon examine properties of perfect numbers and Euler's sigma function. Recall that $\sigma(n)$ is the sum of all the divisors of n (including 1 and n) and that n is perfect if $\sigma(n) = 2n$. (See the article on page 21 for more information.) **The Perfect Problem** asks for two proofs.

1) Prove that if $2^k - 1$ is a prime number and $n = 2^{k-1}$, then $\sigma(\sigma(n)) = 2n$.

2) Prove Hank Sneddon's claim that if n is an even perfect number, then

$$\sigma(\sigma(\sigma(n))) = \sqrt{1+8n} \cdot \sigma(\sqrt{1+8n}).$$

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Problem 313. September's Jungle Gym problem is proposed by Purna Patel and Raymond Viglione of Kean University in Union, New Jersey.

Begin with a parallelogram (such as the blue parallelogram in figure 2). Join the midpoints of the four sides to construct a new parallelogram called the *Varignon parallelogram*. Next, construct a rectangle whose sides are the angle bisectors of the four interior angles of the original parallelogram.

In **Collinear Diagonals,** you are asked to prove that the vertices of the rectangle lie on the diagonals of the Varignon parallelogram.

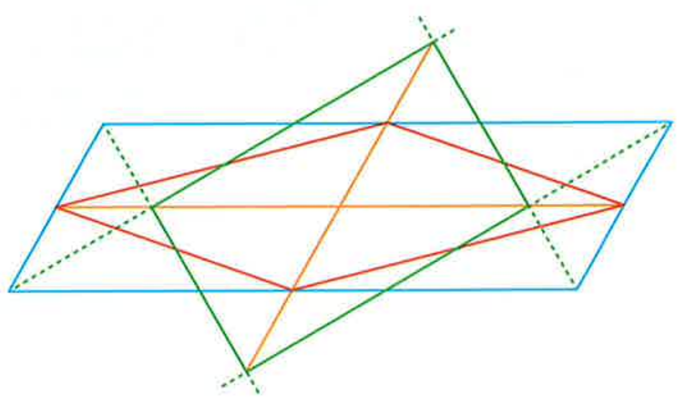


Figure 2.

THE CAROUSEL OLDIES BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Is the surface area of the Earth between the equator and the 45° N latitude line greater than, less than, or equal to the area above the 45° N line? Assume the Earth is a perfect sphere.

FEBRUARY WRAP-UP

In **Problem 302, Digit Sum Series,** we let S_n be the set of all positive integers whose digits sum to n.

a) What are the three smallest integers in the set S_{2014}?

b) Does the sum $\sum_{n \in S_{2014}} \frac{1}{n}$ converge?

We received solutions from Ashley Ward (Westmont College), the Seton Hall University Problem Solving Group, the Northwestern University Math Problem Solving Group, and Dmitry Fleischman. For part a, the three smallest integers in the set are 799...9, 8899...9, and 89899...9, where the number of 9s is chosen so that each number has 224 digits. It's clear that this is the smallest number of digits possible and that these are the three smallest numbers using the minimum number of digits.

The series in part b converges. All solvers used the idea that there are a finite number of ways to write 2,014 as an ordered sum of positive integers. Fix $a_1 + \cdots + a_n = 2{,}014$ as one such sum, where $0 < a_i \leq 9$. Let A_k be the set of all integers in S_{2014} that use the digits $a_1, \ldots, a_n$ in order and have k zeroes. Then A_k has

$$\binom{n+k-1}{k}$$

elements, each of which is greater than 10^{n+k-1}. Then

$$\sum_{m \in A_k} \frac{1}{m} < \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{1}{10^{n+k-1}}.$$

This last series converges by the ratio test, and because there are only a finite number of such ordered sequences, the full series converges.

Problem 303. Yunus Tuncbilek of Ataturk High School of Science in Istanbul, Turkey, posed the **Symmetric Inequality** problem: Assume a, b, and c are positive real numbers. Prove that

$$\frac{1}{9}\left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}\right)^4 + \frac{3abc}{(a+b)(a+c)(b+c)} \geq \frac{15}{16}.$$

We received solutions from all over the planet: Perfetti Paolo (Università degli studi di Tor Vergata Roma); Satyajit Karmakar and Allen Fuller (Gordon State College); Ángel Plaza (Departamento de Matemáticas, Universidad de Las Palmas de Gran Canaria, Spain); Kambiz Razminia (Petroleum University of Technology, Ahwaz, Iran); and Dmitry Fleischman. A partial solution was received from Soham Sinha, a 13-year-old student from White Station High School, Memphis, Tennessee.

The proposer's solution is (briefly) outlined here. First, set $x = \frac{a}{b+c}$, $y = \frac{b}{a+c}$, and $z = \frac{c}{a+b}$. Then the expression on the left becomes

$$\frac{1}{9}(x+y+z)^4 + 3xyz.$$

Yunus then uses the arithmetic-geometric mean inequality and a lot of algebra to show the expression is bounded below by $(1-2xyz)^2 + 3xyz$. Finally,

$$(1-2xyz)^2 + 3xyz - \frac{15}{16} = \frac{1}{16}(8xyz-1)^2 \geq 0.$$

Problem 304. Our February Zip-Line problem, **One Click, Two Click,** came from the article "A 'Mod'ern Mathematics Adventure in *Call of Duty: Black Ops*" by Heidi Hulsizer.

Suppose a safe has three dials; the top dial has digits 0–5, the middle has digits 0–8, and the bottom has positions 0–11. The numbers are in sequential order and the dials turn in only one direction—so that the numbers increase. Turning the top dial one click turns the middle dial two clicks. Turning the middle dial one click turns both the top dial and bottom dials two clicks. And turning the bottom dial one click turns the middle dial two clicks. The dials start at 1, 8, and 2 from top to bottom. To open the safe, the dials must read 3, 6, and 11. Is it possible to open this safe?

We received solutions from Ben Bogard (Wartburg College); Dmitry Fleischman; Thijs Dennison (North Central College); Riley Hall (Westmont College); Anthony J. Pearson, Shane A. Schluter, and Danrun Huang (St. Cloud State University); the Seton Hall University Problem Solving Group; and Bruce Torrence, our former *Math Horizons* editor.

All solvers reduced the problem to solving three equations in three unknowns, either explicitly (often with the help of a computer) or implicitly. An elegant solution came from Riley Hall. Riley reasoned that, because the first and last dials do not affect each other, one can advance the first dial two clicks and the bottom dial nine clicks to correctly situate the top and bottom dials. This advances the middle dial 22 clicks, so it's now set on 3. That means the middle dial needs to be advanced another three clicks to solve the puzzle. But advancing it 12 clicks is the same as advancing it three clicks (because $12 \equiv 3 \mod 9$). Further, this action has no effect on the top or bottom dials because $24 \equiv 0 \mod 6$ implies that the top dial is unchanged, and $24 \equiv 0 \mod 12$ means the same for the bottom dial. This gives a solution: Turn the top dial two clicks, the middle one 12 clicks, and the bottom one nine clicks.

Problem 305. February's Jungle Gym problem, **Archimedes's Revenge,** was from Randy Schwartz of Schoolcraft College (which is in Michigan, not Mississippi).

Archimedes is a young mathematical profiteer who likes to prowl the fairgrounds around Syracuse with a prop called the Magic Globe. Archimedes has written the latitudes and longitudes of two locations on a folded slip of paper.

"Step right up and try your luck!" he bellows to the crowd of onlookers. "Choose your favorite point on the globe, and if the angles of the triangle formed by your point and my two points are all acute, then I win—in all other cases, you win!" The sides of the triangle are simply line segments in three dimensions; Archimedes is not using fancy spherical triangles.

Archimedes knows that if a customer chooses a point at random (so that the probability that a point in a given region is selected is proportional to the area of that region), then the win-lose probabilities depend purely on the central angle between his two points, and there is one such angle that gives Archimedes the highest probability of winning. Find that angle and that probability, and explain why Archimedes decided to rename his prop the Golden Globe.

We received solutions from the Seton Hall University and the Northwestern University problem-solving groups. The proposer's solution is sketched here.

Suppose Archimedes chooses two points with central angle α. By symmetry we may assume they have coordinates $(0,b,c)$ and $(0,b,-c)$. (Figure 3 shows a two-dimensional cross section.) Archimedes loses the bet if the selected point lies in one of three caps: above the plane $z = c$, below $z = -c$, or to the right of $y = b$. Thus, we need to compute the areas of these three spherical caps. We do that using a theorem discovered by Archimedes (the real one): If two parallel planes h units apart slice a sphere of radius r, the surface area between the planes is $2\pi rh$ (this theorem can also be used to solve the Carousel problem!). Using these calculations, we conclude that the probability that Archimedes wins the bet is

$$p(\alpha) = \sin\left(\frac{\alpha}{2}\right) + \frac{1}{2}\left(\cos\left(\frac{\alpha}{2}\right) - 1\right).$$

The maximum of this function occurs when $\alpha = 2\tan^{-1}(2) \approx 126.9^\circ$. If Archimedes picks two points on the sphere with this central angle, the winning probability is

$$p(2\tan^{-1}(2)) = (\sqrt{5} - 1)/2 = \frac{1}{\phi} \approx 0.618,$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio (hence the name the Golden Globe).

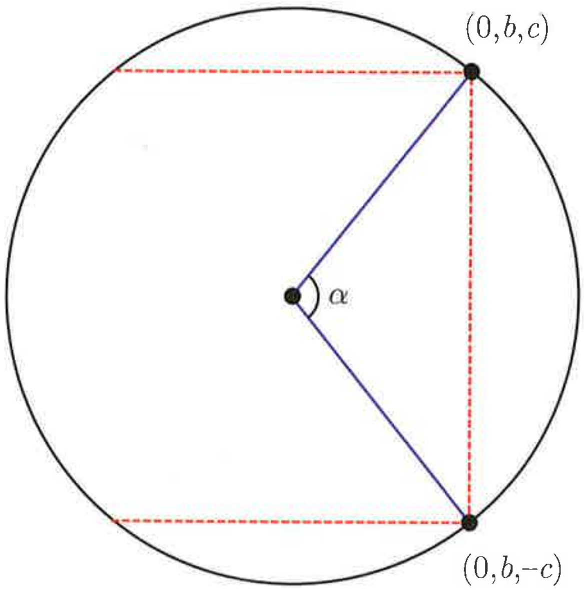


Figure 3.

CAROUSEL SOLUTION

They are equal! You can prove this using the surface area formula from calculus, or you can use Archimedes's theorem (stated in the solution to **Problem 305**) that the surface area of the part of a sphere trapped between two parallel planes depends only on the distance between the two planes.

CLEANING UP

Sarai Mitchel of Westmont College solved Problem 300 about Hamming distance between poker hands, but her solution arrived after our April issue went to press.

Perfetti Paolo points out that Problem 307 in the April 2014 issue also appears as problem 17 in the journal *MathProblems* 3, no. 4, 2013. Editorial comment: These things happen.

Finally, Problem 297 from April 2013 was reincarnated as Problem 1,176 in Stan Wagon's Macalester College Problem of the Week. The problem asked for a finite configuration of points and lines in the plane with each point having a unique "address," where the address of a point is obtained by listing all the k-point lines it's on for all k. (See the description in the April 2013 issue of this column, or the solution given in February 2014.)

Our solution (February 2014) had 16 points. Joseph DeVincentis and Jia Xie each found solutions using only 13 points, and these two solutions are distinct. See figure 4. ■

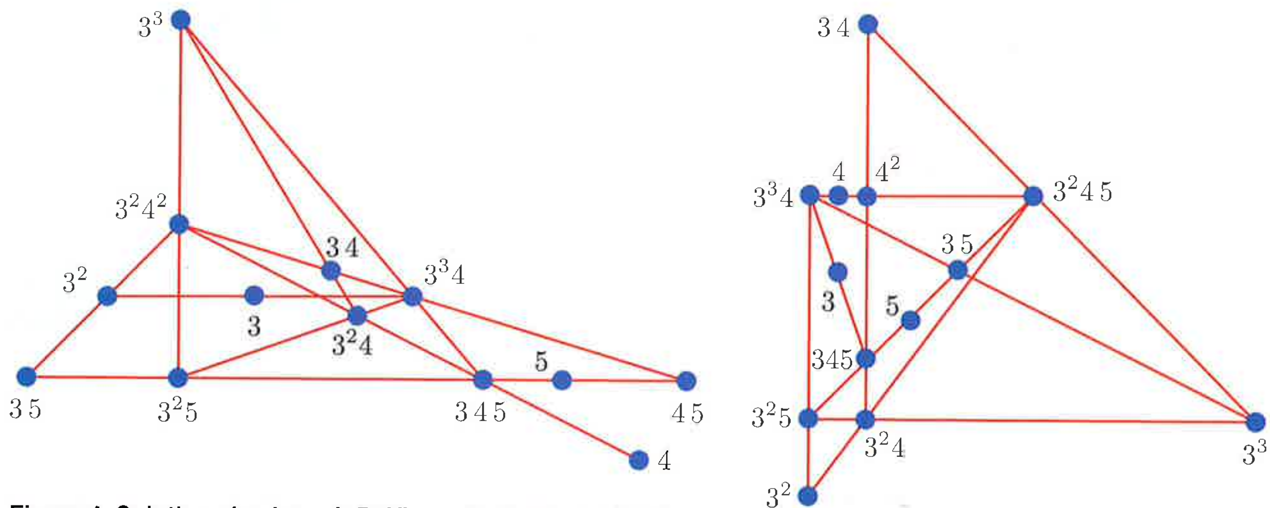
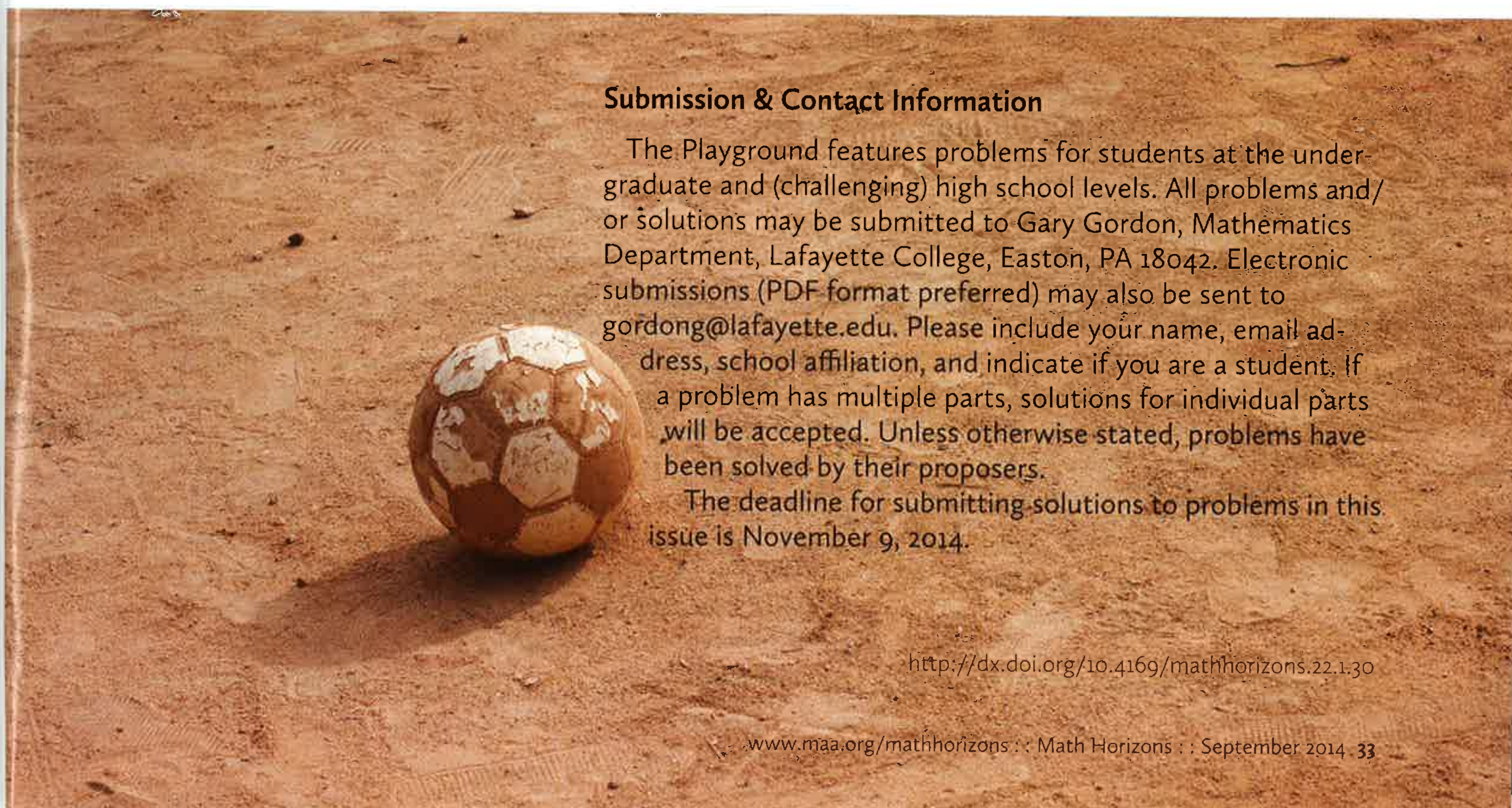


Figure 4. Solutions by Joseph DeVincentis (left) and Jia Xie.



Submission & Contact Information

The Playground features problems for students at the undergraduate and (challenging) high school levels. All problems and/or solutions may be submitted to Gary Gordon, Mathematics Department, Lafayette College, Easton, PA 18042. Electronic submissions (PDF format preferred) may also be sent to gordong@lafayette.edu. Please include your name, email address, school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is November 9, 2014.

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