MATH 145 Final Exam 14 December 2009

Name: ____________________________________

Do any 6 of the following problems (140 points total).

Please circle the problems you want to be graded.
All statements must be proved. Answers without proofs may receive 0 credit.

1. Prove that if all four corner squares are removed from an 8×8 board, then the obtained board cannot be covered with L-tetrominoes.
2. A square is inscribed in a circle with radius 4, and then a smaller circle is inscribed in the square. Find the area of the region (consisting of 4 parts) inside the square but outside the smaller circle.
3. Does the complete bipartite graph $K_{5,10}$ have a Hamilton cycle (a cycle that goes through every vertex exactly once)?
4. Two players play the following game:
   - turns alternate;
   - at each turn, a player removes 1, 2, or 3 counters from a pile that initially contains 100 counters;
   - the player who removes the last counter wins.

   Find a winning strategy for one of the players (clearly identify whether you have to go first or second in order to win, how you will play throughout the game, and show that you will always be able to move as you described).
5. Find an equation of the line with negative slope and passing through the point $(1,1)$ such that the triangle bounded by this line and the axes is divided by the parabola $y = x^2$ into two regions of equal area.
6. We start with the number 4^{2010}. We compute the sum of its digits, and then the sum of the digits of the result, and then again and again until only one digit remains. What is it?
7. Solve the inequality: $3 + x^2 > 6|x - 1|$.
8. Recall that, as discussed in class, n lines in a general position (i.e. no two are parallel and no three have a common point) in a plane divide the plane into $\frac{n^2+n+2}{2}$ regions. Into how many regions do 7 planes in a general position (no two are parallel, no three are parallel to the same line, no four have a common point) divide $\mathbb{R}^3$?

For extra credit (15 points):

- Two non-parallel lines p and q and a point A that does not lie on either of the lines are given. Construct a square $ABCD$ with center O such that vertex C is on line p and point O is on line q.