THE PLAYGROUND

Welcome to the Playground. Playground rules are posted on page 33, except for the most important one: *Have fun!*

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't necessarily mean that they are easy to solve!

Problem 326. In **One-Sum**, Alan Loper (Ohio State, Newark) and Greg Oman (University of Colorado) ask you to find all positive integers n such that the base 10 representation of $1+2+\cdots+n$ consists of all ones.

Problem 327. A two-person game is played as follows. A rectangular box with dimensions $5\times 13\times 31$ is filled with $5\cdot 13\cdot 31 = 2{,}015$ unit cubes, where each unit cube has a unique (x,y,z) address in the box. Alice selects a cube (a,b,c) and removes it; she also removes all the cubes in the three lines through this cube parallel to the edges of the box.

For instance, if Alice chooses (1,2,3), then all cubes with addresses $(x,2,3)$, $(1,y,3)$, and $(1,2,z)$ will be removed. (The remaining cubes stay in place—there is no gravity in this problem.)

Then Bob does the same thing; he chooses a unit cube from what remains, and he removes it and everything else along the three lines determined by his cube. The players alternate turns, each time choosing a cube as above.

Assume each player takes as many cubes as possible each turn. There is no winner; the players hug at the end. **Grabbing Cubes** asks three questions.

1) How long will this game last?
2) How many cubes will each player end up with?
3) Note that 2,015 is the product of three distinct odd primes. Will I still be editing this column the next year this happens?

THE ZIP-LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Problem 328. In *A Conversation with Tim Gowers* (page 10), we discussed Gowers's work with arithmetic progressions. This problem, **Prime Time**, concerns a subsequence of an arithmetic sequence. Show that there is a prime number p and an infinite sequence of positive integers $a_1 < a_2 < \cdots$ so that every term in the sequence $p+2015a_1, p+2015a_2, p+2015a_3, \ldots$ is prime.

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Problem 329. For each positive integer n, draw the quarter-circle $x^2+y^2=n$ in the first quadrant of the xy-plane. Let a_n be the number of positive integer lattice points on this quarter-circle, that is, points (a,b) where both a and b are positive integers. For example, $a_5=2$ because (1,2) and (2,1) satisfy $x^2+y^2=5$. The sequence begins $a_1=0, a_2=1, a_3=0, a_4=0, a_5=2, \ldots$

In **Supernova**, you have two tasks:

1) Show that $a_k=0$ for infinitely many k.
2) Show that $a_k=1$ for infinitely many k.

Extra credit: Can you say anything about the number of times $a_k=n$ for $n>1$?

THE CAROUSEL OLDIES BUT GOODIES

In this section, we present an old problem that we like so much we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Suppose every point in the xy-plane is colored red, green, or blue. Show that there must be two points a distance 1 apart with the same color.

APRIL WRAP-UP

Problem 318. In **Odd Job,** Greg Oman asked you to find all positive integers n and k satisfying

$$1+3+5+\cdots+(2n-1)$$
$$=(2n+1)+(2n+3)+(2n+5)+\cdots+(2n+(2k-1)).$$

We received solutions from Adnan Ali (Mumbai, India), Dmitry Fleischman, Dana Lacey (North Central College), Brooke Logan (Rowan University), Jessop Lueschow (University of Wisconsin, Platteville), Angel Plaza (Universidad de Las Palmas de Gran

Canaria, Spain), Henry Ricardo (New York Math Circle), Tyler Schmitt (University of Wisconsin, Platteville), Randy Schwartz (Schoolcraft College), and the Armstrong Problem Solvers of Armstrong State University, the Missouri State University Problem Solving Group, the Northwestern University Problem Solving Group, the Problem Solving Group from the Department of Financial and Management Engineering (University of the Aegean), the San Francisco University High School Team, and the Pittsburg State University Problem Solving Group.

We also received solutions from two separate teams from Taylor University: Josh Stimmel, Cassidy Wyse, and Becca Gerig; and Jason Kimball, Thaddeus McClatchey, and Cameron Eckman.

There are no solutions to the equation. Note that the sum on the left is n^2, and the sum on the right is $(n+k)^2 - n^2$. Setting these equal and doing a bit of algebra gives $\left(\frac{n+k}{n}\right)^2 = 2$. This is impossible, because, as the ancient Greeks proved before any of us were born, $\sqrt{2}$ is irrational.

Problem 319. Tom Yuster gave us this problem. In **No Roots for You,** Alice and Bob alternate turns, each one assigning a nonzero real number to one of the coefficients of a polynomial, with Alice going first. Once a value has been assigned to a coefficient, it cannot be changed. The game ends when all the coefficients have been assigned. Alice wins if the resulting polynomial has no real roots, and Bob wins otherwise.

1) Suppose Alice and Bob play this game with the quadratic polynomial $ax^2 + bx + c$. Assuming both players use optimal strategies, who will win? Describe that winning strategy.
2) Repeat part 1 for the quartic polynomial $ax^4 + bx^3 + cx^2 + dx + e$.

We received a solution to both parts from the San Francisco University High School Team (C J Dowd, Julia Wei, Benjamin Share-Sapolsky, and Michael Lin).

We also received solutions for part 1 from the Armstrong Problem Solvers (Armstrong State University) and the Problem Solving Group from the Department of Financial and Management Engineering (University of the Aegean), and from two teams from Taylor University: Jason Kimball, Thaddeus McClatchey, and Cameron Eckman; and Josh Stimmel, Cassidy Wyse, and Becca Gerig.

1) Note that Alice wins precisely when the discriminant $b^2 - 4ac < 0$. Alice can force a win by assigning $b = 1$ in her initial turn. When Bob assigns a (nonzero) value to a or c, Alice can select a value for the other coefficient so that $1 - 4ac < 0$ (for example, if Bob assigns a, Alice can choose $c = 1/a$).

2) Bob can always win when the polynomial $p(x)$ is a quartic. We follow the proposer's solution. First, note that Bob wins if a and e have opposite signs. So Alice will choose b, c, or d initially. On Bob's first turn, he sets $a = 1$. This forces Alice to choose a positive value of e on her next turn.

At this point, there are two cases, depending on whether Alice chooses c or one of b or d on her first turn (the choices for b or d can be treated together by symmetry).

Case i: Assume Alice chooses a value for b initially (the d case is identical). Let $f(x) = x^4 + bx^3 + cx^2 + e$, where c is temporarily undetermined. Bob can choose c so that $f(1) < 0$ and $f(-1) < 0$. Now, he wins: No matter what value Alice chooses for d, at least one of $p(1)$ or $p(-1)$ will be negative. Because $p(0) > 0$, p must have a real root.

Case ii: Assume Alice chooses a value for c initially. Then set $f(x) = x^4 + cx^2 + e$. Let M be the absolute maximum of f on the interval $[-2, 2]$. Then Bob can win if he selects $d = -2M$. To see this, note that $g(x) = x^4 + cx^2 - 2Mx + e$ has $g(1) \leq -M$. This forces Alice to choose $b > M$ so that $p(1) > 0$. But then $p(-2) = f(-2) - 8b + 4M < -3M < 0$, so p must have a root.

Tom Yuster points out that this strategy works for all polynomials of degree $2n > 2$. Bob forces the choice of lead and constant terms with the first move. Then Bob removes even degree terms until move n. There will either be an even degree term and an odd left, or two odds. Then Bob plays as above in each case (using the low degree term when there are two odds).

Problem 320. Joshua Bowman's article about billiard paths in polygons gave us **Bank Shot.** There were two parts:

1) A pool table has dimensions $a \times b$ for positive integers a and b, with holes in each of the four corners. A ball is shot from one corner of the table at a 45-degree angle. How many times will it bounce off a wall before it lands in a pocket?

2) Now try this in three dimensions: A box with dimensions $a \times b \times c$ (for positive integers a, b, and c) has holes in each of its eight corners, and a ball is shot from one corner along the line $x = y = z$. How many times will it bounce off a wall before it lands in a pocket?

We received solutions from the Armstrong Problem Solvers of Armstrong State University and the Pittsburg State University Problem Solving Group. We received partial solutions from Dmitry Fleischman, the Missouri State University Problem Solving Group, and the San Francisco University High School Team.

1) The answer is $\frac{a+b}{\gcd(a,b)} - 2$. The easiest procedure uses the *reflection principle.* Instead of having the ball bounce around the table, tile the plane with $a \times b$ rectangles and pretend the path of the ball is the line $y = x$, as in figure 1.

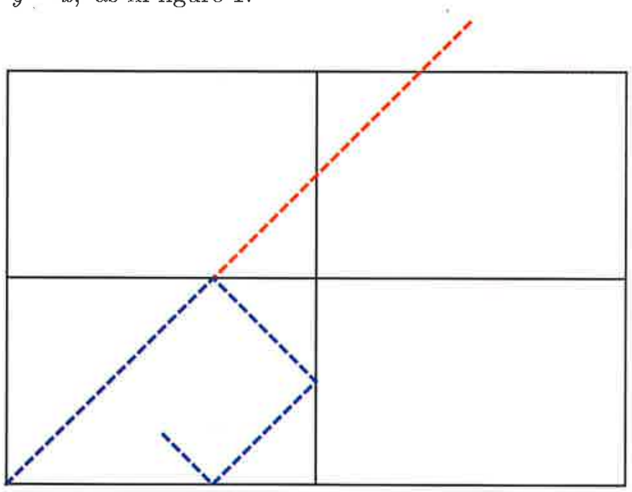

Figure 1.

There is a one-to-one correspondence between the places the ball bounces off a cushion and the places the line $y = x$ crosses one of the grid lines formed by the rectangles. Thus, the ball will land in a pocket when the line $y = x$ first meets a point of the form (ma,nb) for some positive integers m and n. When this happens, the ball will have crossed $m + n - 2$ lines in its path from (0,0) to (ma,nb).

Thus, we seek the smallest values of m and n that satisfy $ma = nb$. That is, $ma = nb = \operatorname{lcm}(a,b)$. Putting this together with the fact that $\operatorname{lcm}(a,b)\gcd(a,b) = ab$, we obtain our answer.

2) The Armstrong Problem Solvers envisioned this as *racquetball in zero gravity.* The reasoning is similar to the solution for part 1. Let $l = \operatorname{lcm}(a,b,c)$, $l_1 = \operatorname{lcm}(a,b)$, $l_2 = \operatorname{lcm}(a,c)$, and $l_3 = \operatorname{lcm}(b,c)$. Then the number of bounces is

$$\frac{l}{a} + \frac{l}{b} + \frac{l}{c} - \left(\frac{l}{l_1} + \frac{l}{l_2} + \frac{l}{l_3}\right).$$

This follows exactly as before, but we need to consider the cases where our ball hits two walls simultaneously. (Note: The incorrect, but tempting, guess of $\frac{l}{a} + \frac{l}{b} + \frac{l}{c} - 3$ is valid when the three numbers are pairwise relatively prime. In the general case, this overcounts the number of bounces and the subtracted terms correct the overcount.)

Problem 321. Draw n lines in the plane, no two parallel, no three (or more) through a common point of intersection. Call a region formed by these lines *trivial* if it is bounded by only two lines. **Region Count** asks you to find the maximum and minimum values for the number of trivial regions n lines can create.

We received complete solutions from the Armstrong Problem Solvers of Armstrong State University and the San Francisco University High School Team. We also received partial solutions from Sarah Seales and the Taylor University team of Josh Stimmel, Cassidy Wyse, and Becca Gerig.

When $n = 2$, there are four trivial regions. When $n > 2$, the minimum number of trivial regions is three and the maximum number is either n (when n is odd) or $n - 1$ (when n is even).

First, the minimum: To see that the minimum must be at least three, choose a circle large enough to contain all of the $\binom{n}{2}$ intersection points. Then shrink the circle until it meets three *extreme* points. Each of those points will be the vertex of a trivial region, as in figure 2.

To see that the minimum is three, the Northwestern group constructed a collection of lines mutually tangent to a circle, as figure 3. It is clear this procedure will create three trivial regions, each one incident to one of the three distinguished points of intersection.

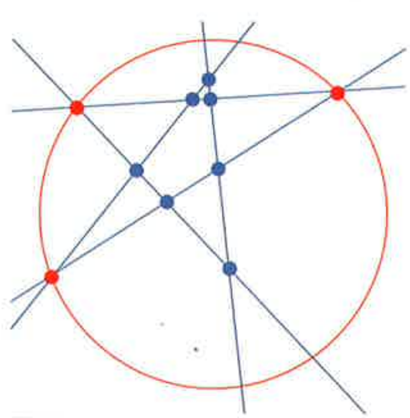

Figure 2.

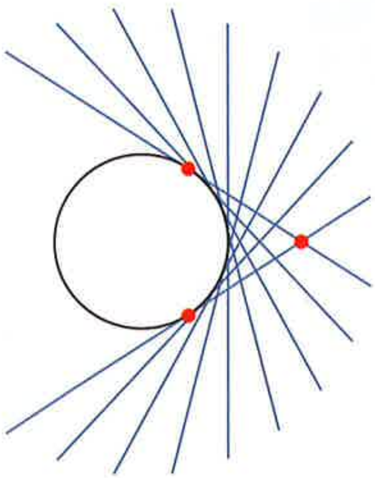

Figure 3.

For the maximum, we again follow the argument given by the Northwestern group. First, note that there are a total of $2n$ unbounded regions. (This can be proven by induction, but it is easy to see: Draw n lines on a piece of paper, then hold the paper very far from your eye. It will appear that the n lines go through a common point, and this clearly gives $2n$ unbounded regions.) It is impossible for trivial regions to share a bounding line; if they did, there would be three lines through a point. Hence, the number of trivial regions is at most n.

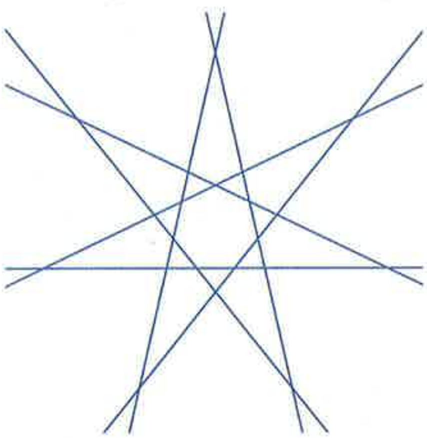

Figure 4.

As before, it remains to show that the maximum can be achieved. For odd n, draw a regular n pointed star, as in figure 4.

For n even, we can achieve $n-1$ regions by first creating a configuration of $n+1$ lines with $n+1$ trivial regions, then removing a line. This destroys two trivial regions, leaving $n-1$. To see this is the largest possible number, suppose there were a configuration with n trivial regions. Then, moving through the $2n$ unbounded regions clockwise, we must encounter trivial and nontrivial regions consecutively. Label these T_1, N_1, T_2, N_2, ..., T_n, N_n, where the T_i are trivial and the N_i are nontrivial. But then $T_{n/2}$ and T_n are trivial regions bounded by the same lines, which is impossible.

CAROUSEL SOLUTION

Suppose we tried to color the plane using three colors so that points one unit apart have different colors. We begin with the seven points in figure 5 called the *Moser spindle* (named after brothers Leo and William Moser). Edges indicate points that are one unit apart.

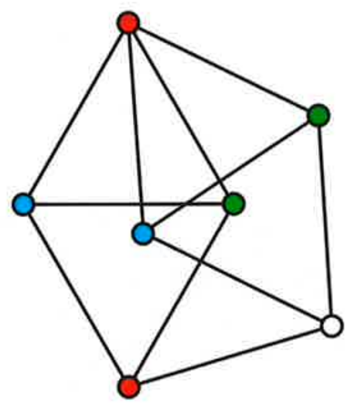

Figure 5.

Color the top point red. The four adjacent points must be blue and green, and one of the remaining points is red, but then there is no color available for the final point. Thus, coloring the plane is impossible.

This is (part of) a famous unsolved problem: Find the minimum number of colors needed to color every point in the plane so that no two points that are a distance 1 apart are the same color. Call this number m. Then the Moser spindle shows $m \geq 4$. For sleepless nights, try showing $m \leq 9$. (In fact, it is known that $4 \leq m \leq 7$.) ■

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Submission & Contact Information

The Playground features problems for students at the undergraduate and (challenging) high school levels. All problems and/or solutions should be submitted to Gary Gordon, Mathematics Department, Lafayette College, Easton, PA 18042. Electronic submissions (PDF format preferred) may also be sent to *gordong@lafayette.edu*. Please include your name, email address, and school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is November 9, 2015.

Corrections

We have two corrections for the April 2015 issue. In "The Spirograph & Mathematical Models from 19th-Century Germany" by Amy Shell-Gellasch, the curtate cycloid was misspelled (as curate cycloid). And in "What Were They Thinking? A Look at Life in 1915" by Deanna Haunsperger and Pamela Richardson Ivan Nivan's first name was misspelled.