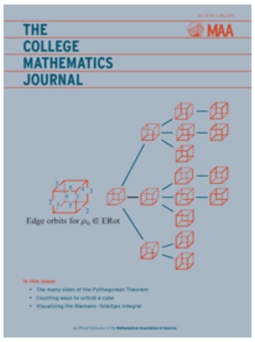

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Problems and Solutions

Greg Oman & Charles N. Curtis

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PROBLEMS AND SOLUTIONS

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This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than November 15, 2019. Sending both pdf and TEXfiles is ideal.

PROBLEMS

1151. *Proposed by Gregory Dresden, Washington and Lee University, Lexington, VA.*

Fix an odd integer b, and set $M = \begin{pmatrix} 1 & b \\ 4 & 5 \end{pmatrix}$. For a positive integer n, let $e(n)$ denote the exponent of the highest power of 2 which divides an entry of M^n. In other words, $2^{e(n)}$ divides some entry in M^n, but no larger power of 2 divides an entry of M^n. Find $e(n)$ as a function of n.

1152. *Yagoub Aliev, ADA University, Azerbaijan.*

Let R be the radius of the circumscribed circle of triangle ABC. Let D be a point on the arc BC, which does not contain A, and drop perpendicular DE to BC. Now take point F on the same arc such that $\angle CAF = 2\angle BAF$. Prove that $8R \cdot \text{Area}(CDE) \leq CF^3$.

1153. *Proposed by Eugen Ionascu, Columbus State University, Columbus, GA.*

Two points B and C are chosen independently at random with uniform distribution on a segment $\overline{AD}$. Show that the probability that A, B, C, and D are equally spaced from some point P not on $\overleftrightarrow{AB}$ is equal to $\mathcal{P} = \frac{15-16\ln 2}{9}$. (By "equally spaced," we mean that the angles $\angle APB$, $\angle BPC$, and $\angle CPD$ have the same measure.)

1154. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Romania.*

For a positive integer n, define $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sin x \sin 2x \sin 3x \cdots \sin nx$. Calculate $f_n^{(n)}(0)$.

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1155. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let F be a field and suppose that K is a field extension of F. If $\alpha \in K$ is transcendental over F, then it is well-known that $F(\alpha) \cong F(x)$, the field of rational functions in x with coefficients in F (here, $F(\alpha)$ denotes the subfield of K generated by F and α). Thus, for example, $\mathbb{Q}(\pi)$ is isomorphic to its proper subfield $\mathbb{Q}(\pi^2)$. Does there exist a field K, which is algebraic over $\mathbb{Q}$ and a proper subfield F of K such that $K \cong F$?

SOLUTIONS

The limit of an infinite product

1126. *Proposed by George Stoica, Saint John, New Brunswick.*

Prove that

$$\lim_{x\to\infty} \prod_{n=1}^{\infty} \left(1 - \frac{1}{x^n}\right) = 1.$$

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY.

Setting $y = \prod_{n=1}^{\infty} \left(1 - x^{-n}\right)$ and using the series expansion

$$\ln(1-u) = -\sum_{n=1}^{\infty} \frac{u^n}{n}, \ |u| < 1,$$

we can write (for $x > 1$)

$$\begin{aligned} \ln y &= \sum_{n=1}^{\infty} \ln\left(1 - x^{-n}\right) \\ &= \sum_{n=1}^{\infty} \left[-\frac{1}{x^n} - O\left(\frac{1}{x^{2n}}\right)\right] \\ &= -\frac{1}{x-1} - O\left(\frac{1}{x^2-1}\right) \to 0 \text{ as } x \to \infty. \end{aligned}$$

Also solved by Ulrich Abel, Mittelhessen, Germany; Robert Agnew, Palm Coast, FL; Michel Bataille, Rouen, France; Paul Budney, Sunderland, MA; Hongwei Chen, Christopher Newport U.; William Cowieson, Fullerton C.; James Duemmel, Bellingham, WA; Bill Dunn, Montgomery C.; Dmitry Fleischman, Santa Monica, CA; Kyle Gatesman, Johns Hopkins U.; Subhankar Gayen, West Bengal, India; GWStat Problem Solving Group; Lixing Han, U. Michigan-Flint; Eugene Herman, Grinnell C.; Yousaf Khan and J. Todd Lee (jointly); Tom Jager, Calvin C.; John Kieffer, U. Minnesota Twin Cities; Jimin Kim, Yonsei U., Seoul, Republic of Korea; Panagiotis Krasopoulos, Athens, Greece; Elias Lampakis, Kiparissia, Greece; Kee-Wai Lau, Hong Kong, China; U. of Louisiana Lafayette Math Club; Northwestern U. Math Problem Solving Group; Paolo Perfetti, U. Roma Tor Vergata, Italy; Henry Ricardo (an additional solution), Westchester Area Math Circle; Nikhil Sahoo (student), Berkeley City C.; Stephen Scheinberg, Corona del Mar, CA; William Seaman, Bethlehem, PA; Ioannis Sfikas, Athens, Greece; Jacob Siehler, Gustavus Adolphus C.; Brendan Sullivan, Emmanuel C.; Shannon Tefft, Ohio Northern U.; Nora Thornber, Raritan Valley Comm. C.; Michael Vowe, Therwil, Switzerland; Lawrence Weill (2 solutions), California St. U., Fullerton; Edward White and Roberta White (jointly), Frostburg, MD; Stuart Witt, Brooklyn, NY; and the proposer.

A limit involving the Ioachimescu constant

1127. *Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.*

For each positive integer n, let $e_n = \left(1+\frac{1}{n}\right)^n$ and $\lim_{n\to\infty} e_n = e$, Euler's constant, and let

$$s_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

and $\lim_{n\to\infty} s_n = s$, the Ioachimescu constant. Find $\lim_{n\to\infty} \frac{e-e_n}{(s_n-s)^2}$.

Solution by William Cowieson, Fullerton College, Fullerton, CA.

Find upper and lower bounds for the numerator and denominator, then use the squeeze theorem to obtain the limit.

Bounds on the numerator: Let $L_n = \log e_n = n\log(1+1/n)$. By the mean value theorem, there is $x_n \in (L_n, 1)$ with

$$e - e_n = e^1 - e^{L_n} = \frac{e^1 - e^{L_n}}{1 - L_n}(1 - L_n) = e^{x_n}(1 - L_n).$$

The alternating series expansion $\log(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}$ reveals that

$$x - \frac{1}{2}x^2 < \log(1+x) < x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \quad \text{for} \quad 0 < x < 1,$$

so let $x = 1/n$, multiply by n, and subtract from 1 to obtain

$$\frac{1}{2n} - \frac{1}{3n^2} < 1 - L_n < \frac{1}{2n}.$$

Clearly $e^{L_n} < e^{x_n} < e^1$, so altogether,

$$\left(1 - \frac{2}{3n}\right)\frac{e_n}{2n} < e - e_n < \frac{e}{2n}.$$

Bounds on the denominator: Note that

$$2\sqrt{n} = \int_0^n x^{-1/2}\,dx = \sum_{k=1}^{n}\int_{k-1}^{k} x^{-1/2}\,dx = \sum_{k=1}^{n} 2(\sqrt{k} - \sqrt{k-1}),$$

so

$$s_n = \sum_{k=1}^{n}\left(2\sqrt{k} - 2\sqrt{k-1} - \frac{1}{\sqrt{k}}\right) \implies s - s_n = \sum_{k=n+1}^{\infty}\left(2\sqrt{k} - 2\sqrt{k-1} - \frac{1}{\sqrt{k}}\right).$$

The kth term in this series simplifies to $1/[\sqrt{k}(\sqrt{k}+\sqrt{k-1})^2]$ which is bounded below by $\frac{1}{4}k^{-3/2}$ and above by $\frac{1}{4}(k-1)^{-3/2}$, so compare with appropriate integrals to obtain

$$\int_{n+1}^{\infty} \frac{1}{4} x^{-3/2}\,dx < \sum_{k=n+1}^{\infty} \frac{1}{4} k^{-3/2} < s - s_n < \sum_{k=n+1}^{\infty} \frac{1}{4}(k-1)^{-3/2} < \int_{n-1}^{\infty} \frac{1}{4} x^{-3/2}\,dx.$$

Finally, evaluate the integrals and square to find

$$\frac{1}{4(n+1)} < (s-s_n)^2 < \frac{1}{4(n-1)}.$$

Find the limit: With the bounds above obtain

$$\begin{aligned} 2e_n\left(1-\frac{1}{n}\right)\left(1-\frac{2}{3n}\right) &= 4(n-1)\left(1-\frac{2}{3n}\right)\frac{e_n}{2n} \\ &< \frac{e-e_n}{(s-s_n)^2} < 4(n+1)\cdot\frac{e}{2n} = 2e\left(1+\frac{1}{n}\right) \end{aligned}$$

and let $n \to \infty$ to find

$$2e = \lim_{n\to\infty} 2e_n\left(1-\frac{1}{n}\right)\left(1-\frac{2}{3n}\right) \le \lim_{n\to\infty} \frac{e-e_n}{(s-s_n)^2} \le \lim_{n\to\infty} 2e\left(1+\frac{1}{n}\right) = 2e.$$

Also solved by Michel Bataille, Rouen, France; Hongwei Chen, Christopher Newport U.; William Cowieson, Fullerton, C.; Dmitry Fleischman, Santa Monica, CA; Kyle Gatesman, Johns Hopkins U.; GWstat Problem Solving Group; Lixing Han, U. Michigan-Flint; Eugene Herman, Grinnell C.; Elias Lampakis, Kiparissia, Greece; U. Louisiana Lafayette Math Club; Moubinool Omarjee, Paris, France; Paolo Perfetti, U. Roma Tor Vergata, Italy; William Seaman, Bethlehem, PA; Ioannis Sfikas, National and Kapodistrian U., Greece; Michael Vowe, Therwil, Switzerland; Edward White and Roberta White (jointly), Frostburg, MD; and the proposer.

Fractions

1128. *Proposed by Arthur L. Holshouser, Charlotte, NC, and Benjamin G. Klein, Davidson College, Davidson, NC.*

Let a, b, c, d be positive integers such that a and b are relatively prime, similarly c and d, and $\frac{a}{b} < \frac{c}{d}$. Find necessary and sufficient conditions on a, b, c, d such that if x and y are relatively prime positive integers with $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$, then $u = -dx + cy$ and $v = bx - ay$ are relatively prime.

Solution by Seungjae Yun, Seoul Science High School, Seoul, Korea.

We claim that $bc - ad = 1$ is a necessary and sufficient condition. In what follows, (a, b) denotes the greatest common divisor.

First, if $bc - ad = 1$, then following holds:

$$\begin{aligned} (-dx+cy, bx-ay) &\mid (-dx+cy, (bx-ay)d + (-dx+cy)b) \\ &= (-dx+cy, (bc-ad)y) \end{aligned}$$

Since $-dx+cy>0$ and $bc-ad>0$,

$$(-dx+cy,(bc-ad)y)=(-dx+cy,y)=(d,y).$$

Thus

$$(-dx+cy,bx-ay)\mid(d,y) \tag{1}$$

In the same way,

$$(-dx+cy,bx-ay)\mid(c,x). \tag{2}$$

By (1) and (2), we find that $(-dx+cy,bx-ay)=1$, since $(-dx+cy,bx-ay)$ is a divisor of (d,y) and (c,x) but $(x,y)=1$.

Now, let's assume that $(x,y)=1$ and $\frac{a}{b}<\frac{x}{y}<\frac{c}{d}$ implies that $(-dx+cy,bx-ay)=1$. Let x and y be integers that satisfy

$$\frac{x}{y}=\frac{a+c}{b+d} \quad \& \quad (x,y)=1.$$

Then, since $\frac{a}{b}<\frac{c}{d}$, $\frac{a}{b}<\frac{x}{y}<\frac{c}{d}$ holds, substituting $a+c$ for x and $b+d$ for y gives

$$\begin{aligned}1&=(-dx+cy,bx-ay)=(-d(a+c)+c(b+d),b(a+c)-a(b+d))\\&=bc-ad.\end{aligned}$$

Thus $bc-ad=1$ holds.

Also solved by John Christopher, California St. U., Sacramento; James Duemmel, Bellingham, WA; Dmitry Fleischman, Santa Monica, CA; Abhay Goel (student), Kalamazoo C.; Jimin Kim, Younsei U., Seoul, Republic of Korea; William Seaman, Bethlehem, PA; Jacob Siehler, Gustavus Adolphus C. Lucas Stefanic (student) Rochester Inst. of Tech.; Brendan Sullivan, Emmanuel C.; and the proposer.

A characterization of the natural logarithm

1129. *Proposed by David Bradley, University of Maine, Orono, ME.*

The natural logarithm satisfies the functional equation $\log(xy)=\log(x)+\log(y)$ for positive real x and y. It also satisfies the inequality $\log(x)\le x-1$ for positive real x. Show that these two properties characterize the natural logarithm. That is, if the function $f:\mathcal{R}_{>0}\to\mathcal{R}$ satisfies the functional equation $f(xy)=f(x)+f(y)$ and the inequality $f(x)\le x-1$ for all positive real x and y, then $f(x)=\log(x)$ for all positive real x.

Solution by Dmitry Fleischman, Santa Monica, CA.

Replacing x and y both with 1 in the functional equation gives $f(1)=0$. For $x>0$ and $y=1/x$, the equation gives $f(x)+f(1/x)=f(1)=0$, so that $f(1/x)=-f(x)$. The inequality implies that for $0<x<1$, $f(x)<0$; hence, for $x>1$, $f(x)>0$.

By induction, $f\left(x^k\right)=kf(x)$ for all positive integers k, from which another induction proves that for positive integers k and m, $f\left(x^{k/m}\right)=\frac{k}{m}f(x)$.

Now suppose that $x > 1$ and r is a positive rational number. Then $x^r > 1$, so that

$$rf(x) = f(x^r) \le x^r - 1,$$

implying that $f(x) \le \frac{x^r-1}{r}$. As r approaches 0, the right-hand side approaches $\ln x$.

Similarly,

$$-rf(x) = f\left(\frac{1}{x^r}\right) \le \left(\frac{1}{x}\right)^r - 1,$$

implying that $f(x) \ge \frac{1-\left(\frac{1}{x}\right)^r}{r}$. As r approaches 0, the right-hand side approaches $\ln x$.

Hence $f(x) \le \ln x$ and $f(x) \ge \ln x$, so $f(x) = \ln x$.

Also solved by ADA U. PROBLEM SOLVING GROUP; MICHEL BATAILLE, Rouen, France; JAMES CARPENTER, Iona C.; WILLIAM COWIESON, Fullerton C.; RICHARD DAQUILA, Muskingum U.; WENWEN DU, Glenville St. C.; JAMES DUEMMEL, Bellingham, WA; KYLE GATESMAN, Johns Hopkins U.; SAM GLICK and TODD LEE (jointly), Elon U.; ABHAY GOEL (student), Kalamazoo C.; LIXING HAN, U. Michigan-Flint; DON HANCOCK, Pepperdine U.; EUGENE HERMAN, Grinnell C.; TOM JAGER, Calvin C.; NATHAN JOHNSON, Rochester Inst. of Tech; JOHN KIEFFER, U. Minnesota Twin Cities; MOHAMMED KHARBACH, ADNOC Gas Processing - OPP; ELIAS LAMPAKIS, Kiparissia, Greece; LUIS MORENO, SUNY Broome Comm. C.; NORTHWESTERN U. MATH PROBLEM SOLVING GROUP; MOUBINOOL OMARJEE, Paris, France; ARTHUR ROSENTHAL, Salem St. U.; WILLIAM SEAMAN, Bethlehem, PA; IOANNIS SFIKAS, Athens, Greece; SKIDMORE C. PROBLEM GROUP; BRENDAN SULLIVAN, Emmanuel C.; TEXAS ST. U. PROBLEM SOLVING GROUP; NORA THORNBER, Raritan Valley Comm. C.; ANNE VAKARIETIS, Culinary Inst. of Virginia; MICHAEL VOWE, Therwil, Switzerland; LAWRENCE WEILL, California St. U., Fullerton; EDWARD WHITE and Roberta White (jointly), Frostburg, MD; MICHAEL WOODRUFF (student), Weber St. U.; JOHN ZACHARIAS, Alexandria, VA; and the proposer.

A sequence of triangles

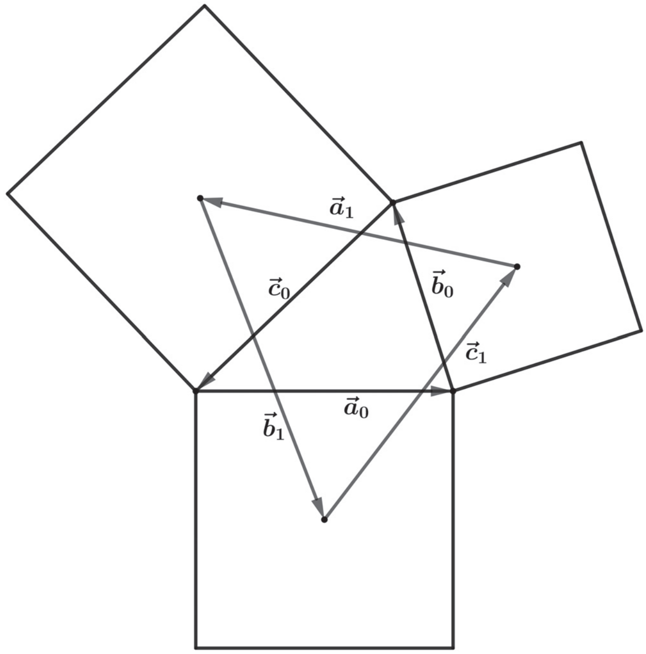
1130. *Proposed by Michael Goldberg, Baltimore Polytechnic Institute, Baltimore, MD, and Mark Kaplan, Towson University, Towson, MD.* Let T_0 be an arbitrary triangle with vertices A_0, B_0, C_0 and corresponding side lengths a_0, b_0, c_0. Construct triangle T_1 whose vertices A_1, B_1, C_1 are the centers of the squares constructed on the sides A_0B_0, B_0C_0, C_0A_0 outside T_0, respectively; call the corresponding side lengths a_1, b_1, c_1. Continue in this way to build triangles T_2, T_3, etc. Show that there exists $\gamma > 0$ such that there exist finite nonzero limits

$$\lim_{n\to\infty} \frac{a_n}{\gamma_n} = \lim_{n\to\infty} \frac{b_n}{\gamma_n} = \lim_{n\to\infty} \frac{c_n}{\gamma_n},$$

that is, such that the sequence of triangles $\{T_n$ scaled by $1/\gamma^n\}$ converges to an equilateral triangle.

Solution by Math Club of the University of Louisiana at Lafayette.

Let $\vec{a}_n$, $\vec{b}_n$, $\vec{c}_n$ denote the sides of the triangle T_n represented as vectors turning counterclockwise (so $\vec{a}_n + \vec{b}_n + \vec{c}_n = 0$) (see figure).



We regard $\vec{a}_n, \vec{b}_n, \vec{c}_n$ as complex numbers. Let $\eta = \frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}} = \frac{1+i}{2}$. Then, inspecting the figure we deduce that

$$\vec{a}_1 = \eta \vec{b}_0 + \overline{\eta} \vec{c}_0.$$

Similar equations hold for $\vec{b}_1$ and $\vec{c}_1$. These can be put in the form

$$\begin{pmatrix} 0 & \eta & \bar{\eta} \\ \bar{\eta} & 0 & \eta \\ \eta & \bar{\eta} & 0 \end{pmatrix} \begin{pmatrix} \vec{a}_0 \\ \vec{b}_0 \\ \vec{c}_0 \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \\ \vec{b}_1 \\ \vec{c}_1 \end{pmatrix}.$$

Let $v_n = (\vec{a}_n, \vec{b}_n, \vec{c}_n) \in \mathbb{C}^3$ for $n \geq 0$. Let A denote the matrix on the left side of the equation above. Then, $v_n = A^n v_0$.

Since A is selfadjoint, it is diagonalizable. Its eigenvalues can be easily calculated since one eigenvector is guessable: (1, 1, 1) is an eigenvector with eigenvalue 1. Set $\chi = e^{2\pi i/3} = \frac{-1+\sqrt{3}i}{2}$. Here is a list of eigenvalues and eigenvectors per eigenvalue:

$$\lambda_1 = 1, \quad w_1 = (1, 1, 1),$$

$$\lambda_2 = \frac{-1+\sqrt{3}}{2}, \quad w_2 = (\chi^2, \chi, 1),$$

$$\lambda_3 = \frac{-1-\sqrt{3}}{2}, \quad w_3 = (\chi, \chi^2, 1).$$

The eigenvalue λ_3 is the largest in absolute value. We claim that $\frac{1}{\lambda_3^n} v_n$ tends to a nonzero scalar multiple of w_3. Proof: Write $A = WDW^*$, where D is diagonal with diagonal $(\lambda_1, \lambda_2, \lambda_3)$ and W is a unitary matrix. Then

$$\frac{1}{\lambda_3^n} v_n = \frac{1}{\lambda_3^n} A^n v_0 = W\Big(\frac{1}{\lambda_3^n} D^n\Big) W^* v_0 \to \langle v_0, w_3 \rangle w_3.$$

Let us argue that the scalar $\langle v_0, w_3\rangle$ is nonzero, that is, v_0 is not orthogonal to w_3. Since $v_0 = (\vec{a}_0, \vec{b}_0, \vec{c}_0)$ and $\vec{a}_0 + \vec{b}_0 + \vec{c}_0 = 0$, v_0 is orthogonal to $w_1 = (1, 1, 1)$. If v_0 were also orthogonal to w_3, it would be a scalar multiple of w_2 (since $\{w_1, w_2, w_3\}$ form an orthogonal basis of $\mathbb{C}^3$). But the entries of v_0 turn counterclockwise while the entries of $w_2 = (\chi^2, \chi, 1)$ turn clockwise (and this is also the case for any scalar multiple of w_2, i.e., for $(z\chi^2, z\chi, z)$ with $z \in \mathbb{C}$). Thus, v_0 is not orthogonal to w_3.

Observe that the entries of w_3 are roots of unity of order 3 and so have absolute value 1. Thus, taking absolute value entrywise in $\frac{1}{\lambda_3^n} v_n \to w_3\langle v, w_3\rangle$ we get that

$$\lim \frac{a_n}{\gamma^n} = \lim \frac{b_n}{\gamma^n} = \lim \frac{c_n}{\gamma^n} \neq 0,$$

where $\gamma = |\lambda_3| = (1 + \sqrt{3})/2$.

Also solved by William Cowieson, Fullerton C.; Kyle Gatesman, Johns Hopkins U.; Jeffrey Groah, Lone Star C.- Montgomery; Eugene Herman, Grinnell C.; Elias Lampakis, Kiparissia, Greece; Lawrence Weill, California St. U., Fullerton; John Zacharias, Alexandria, VA; and the proposer.

Editorial Correction

The January 2019 Problems and Solutions column misattributed some of the proposed problems to the wrong authors.

- Problem 1142 should have been attributed to Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.
- Problem 1143 should have been attributed to George Stoica, Saint John, New Brunswick, Canada.
- Problem 1144 should have been attributed to Andrew Wu, St. Albans School, McLean, VA.

The Editors deeply regret this mistake, and thank the authors listed above for their valuable contributions to *The College Mathematics Journal*.