The College Mathematics Journal

ISSN: 0746-8342 (Print) 1931-1346 (Online) Journal homepage: https://maa.tandfonline.com/loi/ucmj20

Problems and Solutions

Greg Oman & Charles N. Curtis

To cite this article: Greg Oman & Charles N. Curtis (2019) Problems and Solutions, The College Mathematics Journal, 50:5, 379-384, DOI: 10.1080/07468342.2019.1667205

To link to this article: https://doi.org/10.1080/07468342.2019.1667205

Published online: 15 Nov 2019.

PROBLEMS AND SOLUTIONS

EDITORS

Greg Oman
CMJ Problems
Department of Mathematics
University of Colorado, Colorado Springs
1425 Austin Bluffs Parkway
Colorado Springs, CO 80918
email: cmjproblems@maa.org

Charles N. Curtis
CMJ Solutions
Mathematics Department
Missouri Southern State University
3950 E Newman Road
Joplin, MO 64801
email: cmjsolutions@maa.org

This section contains problems intended to challenge students and teachers of college mathematics. We urge you to participate actively *both* by submitting solutions and by proposing problems that are new and interesting. To promote variety, the editors welcome problem proposals that span the entire undergraduate curriculum.

Proposed problems should be sent to **Greg Oman**, either by email (preferred) as a pdf, TEX, or Word attachment or by mail to the address provided above. Whenever possible, a proposed problem should be accompanied by a solution, appropriate references, and any other material that would be helpful to the editors. Proposers should submit problems only if the proposed problem is not under consideration by another journal.

Solutions to the problems in this issue should be sent to **Chip Curtis**, either by email as a pdf, TEX, or Word attachment (preferred) or by mail to the address provided above, no later than May 15, 2020. Sending both pdf and TEXfiles is ideal.

PROBLEMS

1161. *Proposed by Jathan Austin, Salisbury University, Salisbury, MD.*

A *primitive Pythagorean triple* is an ordered triple of positive integers (a, b, c) such that $a^2 + b^2 = c^2$ and $\gcd(a, b, c) = 1$. Let (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) be distinct primitive Pythagorean triples. Prove that, as vectors in $\mathbb{R}^3$, these triples form a linearly independent set.

1162. *Proposed by Dennis S. Bernstein, University of Michigan, Ann Arbor, MI, Adam Bruce, University of Michigan, Ann Arbor, MI, and Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.*

Let $a, b \in (0, 1)$, let c be a positive real number, and assume that the following inequality holds:

$$\sqrt{(a-b)^2 + c} < 1 - a.$$

Prove that, moreover, we have $\sqrt{(a-b)^2 + c} < (1-a)\sqrt{b^2 + c} + a(1-b)$.

1163. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following equation for all real x:

$$f(-x) = 1 + \int_0^x \sin(t) f(x-t)\,dt.$$

doi.org/10.1080/07468342.2019.1667205

1164. *Proposed by George Stoica, New Brunswick, Canada.*

Let $c > 0$ and $f : [0, \infty) \to [0, \infty)$ be Riemann integrable on every bounded interval in $[0, \infty)$. If

$$f(x) \le \frac{1}{c} \int_0^1 \left(\int_0^x f(ts)ds \right) dt$$

for all real $x \ge 0$, prove that $f(x) = 0$ for all $x \in [0, \infty)$.

1165. *Proposed by George Stoica, New Brunswick, Canada.*

Let $a, x_1, \ldots, x_n \in \mathbb{C}$. Prove that $\sum_{j=1}^n x_j^k = a$ for all $k \in \{1, \ldots, n\}$ if and only if $\sum_{j=0}^n (-1)^j \binom{a}{j} x_k^{n-j} = 0$ for all $k \in \{1, \ldots, n\}$.

SOLUTIONS

A condition on commutators that implies a group is abelian

1136. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let G be a (multiplicative) group. For $a, b \in G$, let $[a, b] = a^{-1}b^{-1}ab$. Assume there is a fixed positive integer n such that $[x^n, y] = [x, y^{n+1}]$ for all $x, y \in G$. Prove that G is abelian.

Solution by the proposer.

We use exponential notation $x^y = y^{-1}xy$ for conjugation in G, so that

$$[x, y] = x^{-1}x^y \text{ and } [x, y^{-1}] = y^x y^{-1}. \tag{1}$$

We make use of the following commutator identities

$$[xy, z] = [x, z]^y [y, z] \text{ and } [x, yz] = [x, z][x, y]^z, \tag{2}$$

which can be checked by direct computation.

Replacing x by xy in the hypothesis and using $[y, y^{n+1}] = 1$ (where 1 is the unit of the group G), we have

$$[(xy)^n, y] = [xy, y^{n+1}] = [x, y^{n+1}]^y = [x^n, y]^y. \tag{3}$$

Replacing y bu $x^{-1}y$ in the two expressions on the extremes of (3), we obtain

$$[y^n, x^{-1}y] = [x^n, x^{-1}y]^{x^{-1}y}. \tag{4}$$

The left-hand side of (4) is $[y^n, x^{-1}]^y$ by (2). For similar reasons, the right-hand side of (4) is $[x^n, y]^{x^{-1}y}$. Hence, upon conjugating each side by y^{-1} and using the hypothesis, we have

$$[y^n, x^{-1}] = [x^n, y]^{x^{-1}} = [x, y^{n+1}]^{x^{-1}}. \tag{5}$$

The left-hand side of (5) is $x^{y^n}x^{-1}$, while the right-hand side of (5) is $(x^{-1}x^{y^{n+1}})^{x^{-1}} = x^{y^{n+1}}x^{-1}$ by (1). Therefore,

$$x^{y^n} = x^{y^{n+1}},$$

and hence $x = x^y$ (after conjugating by y^{-n}), i.e., $yx = xy$. This completes the proof.

No other solutions were received.

The limit of a sequence involving sines

1137. *Proposed by D.M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.*

If $a \in (0,1)$ and $b = \arcsin a$, calculate $\lim_{n\to\infty} \sqrt[n]{n!}\left(\sin\left(\frac{b\cdot\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}\right) - a\right)$.

Solution by Theo Koupelis, Broward College, Pembroke Pines, Florida.

From Stirling's approximation we know that $\sqrt[n]{n!} \approx (2\pi n)^{\frac{1}{2n}} \cdot \frac{n}{e}$. Setting $x = 2\pi n$, $y = \frac{1}{2n} < 1$, and using $x^y = e^{y\ln x}$, and the expansion $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, (with $-\infty < z < \infty$), we get $\lim\limits_{n\to\infty} \sqrt[n]{n!} = \lim\limits_{n\to\infty}\left[1 + O\left(\frac{\ln(2\pi n)}{2n}\right)\right] \cdot \dfrac{n}{e}$, and

$$\begin{aligned}\lim_{n\to\infty} \frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} &= \lim_{n\to\infty} (2\pi n)^{-\frac{1}{2n(n+1)}}\left(1+\frac{1}{n}\right)^{1+\frac{1}{2(n+1)}} \\ &= \lim_{n\to\infty}\left[1 - O\left(\frac{\ln(2\pi n)}{2n(n+1)}\right)\right]\left[1 + \frac{1}{n} + O\left(\frac{1}{2n(n+1)}\right)\right].\end{aligned}$$

Therefore, with $a = \sin b$, and $\sin(2\theta) - \sin(2\phi) = 2\sin(\theta - \phi)\cos(\theta+\phi)$, the given limit is equivalent to

$$\lim_{n\to\infty} \sqrt[n]{n!}\cdot 2\sin\left[\frac{1}{2}b\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} - 1\right)\right]\cos\left[\frac{1}{2}b\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} + 1\right)\right],$$

or

$$\lim_{n\to\infty} \frac{n}{e}\cdot 2\sin\left(\frac{b}{2n}\right)\cos b = \lim_{n\to\infty} \frac{b}{e}\cdot\frac{\sin(\frac{b}{2n})}{\frac{b}{2n}}\cdot\cos b = \frac{b\cdot\cos b}{e},$$

because $\lim\limits_{t\to 0} \frac{\sin t}{t} = 1$.

Also solved by Michel Bataille, Rouen, France; Paul Bracken, U. of Texas, Edinburgh; Brian Bradie, Christopher Newport U.; Hongwei Chen, Christopher Newport U.; William Cowieson, Fullerton C.; James Duemmel, Bellingham, WA; Dmitry Fleischman, Santa Monica, CA; Subhankar Gayen, West Bengal, India; Michael Goldenberg, Baltimore Polytechnic Inst. and Mark Kaplan, Towson U. (jointly); The Iowa State Undergraduate Problem Solving Group; Elias Lampakis, Kiparissia, Greece; Ioana Mihăilă, Cal Poly Pomona; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Randy Schwartz, Schoolcraft C.; Nora Thornber, Raritan Valley Comm. C.; Michael Vowe, Therwil, Switzerland; and the proposer. One incorrect solution was received.

A condition for a ring to be a principal ideal ring

1138. *Proposed by Souvik Dey, Ph.D. student, University of Kansas, Lawrence, KS.*

Let R be a finite commutative ring with identity such that distinct ideals of R have distinct cardinalities. Prove that R is a principal ideal ring, that is, every ideal of R is principal.

Solution by Anthony Bevelacqua, University of North Dakota.

We will show that, *without assuming the ring is commutative*, every finite ring with identity such that distinct ideals have distinct cardinalities is a principal ideal ring.

We note that if R is a finite ring with identity such that distinct ideals of R have distinct cardinalities then the same is true in any homomorphic image of R. Denote the principal ideal generated by $z \in R$ by

$$(z) = \{r_1 z s_1 + \cdots + r_n z s_n \mid r_i, s_i \in R \text{ for any } n \geq 1\}.$$

Assume that not every such ring is a principal ideal ring. Let R be a counter-example of least cardinality, and let I be a non-principal ideal of R. Since the zero ideal $0 = (0_R)$ and $R = (1_R)$ are principal ideals, I must be a non-trivial ideal of R. Let $J \subseteq I$ be a minimal ideal of R, that is, $J \neq 0$ and no smaller non-zero ideal of R lives in J. So, for any $0 \neq a \in J$, we have $J = (a)$. Now R/J is a principal ideal ring since $|R/J| < |R|$. Thus there is an $x \in R$ with $I/J = ((x) + J)/J$. Therefore $I = (x) + (a)$. Note also that $I = (x + a) + (a)$.

Assume that both $(x) \cap (a) = 0$ and $(x + a) \cap (a) = 0$. We have $|I/J| = |(x)|$ because

$$I/J = \frac{(x) + (a)}{(a)} \cong \frac{(x)}{(x) \cap (a)} \cong (x).$$

Similarly, $|I/J| = |(x + a)|$. Therefore $(x) = (x + a)$. Thus $x + a \in (x)$ and so $a \in (x)$, a contradiction with $(x) \cap (a) = 0$.

Now for some $y \in \{x, x + a\}$ we have

$$I = (y) + (a) \text{ and } (y) \cap (a) \neq 0.$$

Since (a) is a minimal ideal, we must have $(y) \cap (a) = (a)$, so $(a) \subseteq (y)$. Thus $I = (y)$, a contradiction with I is non-principal.

Therefore, every finite ring with identity such that distinct ideals have distinct cardinalities is a principal ideal ring.

Also solved by Brendan Dosch, North Central C.; Ioannis Sfikas, Athens, Greece; and the proposer.

A sum involving Stirling numbers

1139. *Proposed by Paul Bracken, University of Texas, Edinburg, TX.*

(a) Prove that for any positive integer n, $\int_0^1 \frac{(\log(1-t))^n}{t} dt = (-1)^n n! \zeta(n + 1)$, where ζ denotes the Riemann zeta function.

(b) Sum the following series in closed form, where n is a fixed positive integer:

$$\sum_{p=n}^{\infty} (-1)^p \frac{s(p, n)}{p \cdot (p!)},$$

where the $s(p, n)$ are Stirling numbers of the first kind.

Solution by Hongwei Chen, Christopher Newport University.

(a) For nonnegative integers k, n, integrating by parts gives

$$\int_0^1 x^k (\ln x)^n \, dx = -\frac{n}{k+1} \int_0^1 x^k (\ln x)^{n-1} \, dx,$$

Repeatedly applying this formula finally yields

$$\int_0^1 x^k (\ln x)^n \, dx = \frac{(-1)^n \, n!}{(k+1)^{n+1}}.$$

This implies that

$$\begin{aligned}
\int_0^1 \frac{(\ln(1-t))^n}{t} \, dt &= \int_0^1 \frac{(\ln x)^n}{1-x} \, dx \qquad (\text{let } t = 1-x) \\
&= \int_0^1 \left(\sum_{k=0}^{\infty} x^k \right) (\ln x)^n \, dx \qquad (\text{using the geometric series}) \\
&= \sum_{k=0}^{\infty} \int_0^1 x^k (\ln x)^n \, dx = (-1)^n \, n! \sum_{k=0}^{\infty} \frac{1}{(k+1)^{n+1}} \\
&= (-1)^n \, n! \zeta(n+1),
\end{aligned}$$

which proves (a).

(b) We first show that

$$\sum_{p=n}^{\infty} s(p,n) \frac{x^p}{p!} = \frac{1}{n!} (\ln(1+x))^n. \tag{1}$$

To see this, we compute $(1+x)^t$ in two different ways. By the binomial theorem, in view of the definition of the Stirling numbers, we have

$$(1+x)^t = \sum_{p=0}^{\infty} \binom{t}{p} x^p = \sum_{p=0}^{\infty} \frac{x^p}{p!} \left(\sum_{n=0}^{p} s(p,n) t^n \right) = \sum_{n=0}^{\infty} t^n \left(\sum_{p=n}^{\infty} s(p,n) \frac{x^p}{p!} \right). \tag{2}$$

On the other hand, we have

$$(1+x)^t = \exp(\ln(1+x)t) = \sum_{n=0}^{\infty} \frac{(\ln(1+x))^n}{n!} t^n. \tag{3}$$

Now (1) follows from (2) and (3) by matching the coefficients of t^n. Letting $x = -t$ in (1) gives

$$\sum_{p=n}^{\infty} (-1)^p s(p,n) \frac{t^p}{p!} = \frac{1}{n!} (\ln(1-t))^n.$$

Dividing this equation by t on both sides, then integrating with respect t on [0, 1], from the result of (a) we find that

$$\sum_{p=n}^{\infty}(-1)^p\frac{s(p,n)}{p(p!)}=\frac{1}{n!}\int_0^1\frac{(\ln(1-t))^n}{t}\,dt=(-1)^n\zeta(n+1).$$

Also solved by Ulrich Abel, Technische Hochschule, Mittelhessen, Germany; Michel Bataille, Rouen, France; Khristo Boyadzhiev, Ohio Northern U.; Brian Bradie, Christopher Newport U.; Peter Clark and Isaac Wass (students, jointly), Iowa St. U.; Dmitry Fleischman, Santa Monica, CA (part (a) only); Subhankar Gayen, West Bengal, India (part (a) only); J. A. Grzesik, Allwave Corp.; Henry Ricardo, Westchester Area Math Circle (two solutions); Michael Vowe, Therwil, Switzerland; A. David Wuncsh, U. Mass Lowell (part (a) only); and the proposer.

Uncountable ordinals

1140. *Proposed by Greg Oman, University of Colorado at Colorado Springs, Colorado Springs, CO.*

Let Ω denote the first uncountable ordinal. For the purposes of this problem, this means there is a well-order $<$ on the uncountable set Ω with the property that for every $i \in \Omega$, we have $\text{seg}(i) := \{x \in \Omega\colon x \leq i\}$ is countable. Suppose now that $(r_i : i \in \Omega)$ is a real-valued sequence indexed by Ω such that $r_i \neq r_j$ for $i \neq j$. Next, let $i, j \in \Omega$. Say that r_j is a *future right neighbor* of r_i provided $j > i$ and $r_i < r_j$. Similarly, r_j is a *future left neighbor* of r_i provided $j > i$ and $r_j < r_i$. Prove that $\mathcal{S} := \{i \in \Omega\colon r_i$ has both a future left neighbor and future right neighbor$\}$ is a co-countable subset of Ω, that is, $\Omega\backslash\mathcal{S}$ is countable.

Solution by Nikhil Sahoo (student) University of California-Berkeley.

Define

$$L = \{x \in \Omega : x \text{ has no future right neighbor}\}$$

and

$$R = \{x \in \Omega : x \text{ has no future left neighbor}\}.$$

Then $L \cup R = \Omega \setminus S$. We assume for the sake of contradiction that $\Omega \setminus S$ is uncountable. Then either L or R is uncountable. We may assume without loss of generality that R is uncountable. As a subset of Ω, the set R is also well-ordered. If R has a greatest element x, then $R \subset \text{seg}(x) = \{y \in \Omega : y \leq x\}$, which contradicts the uncountability of R. Thus R is a limit ordinal, so we let $s : R \to R$ be the successor function on R. For any $x, y \in R$ with $x < y$, we have $r_x < r_y$ by definition of R. For each $x \in R$, we choose some $q_x \in \mathbb{Q} \cap \left(r_x, r_{s(x)}\right)$. This defines an injective function $R \hookrightarrow \mathbb{Q}$, contradicting the countability of $\mathbb{Q}$. Therefore we conclude that $\Omega \setminus S$ is countable, i.e., S is co-countable.

Also solved by William Cowieson, Fullerton C.; Northwestern U. Math Problem Solving Group; and the proposer.

Correction: William Seaman has pointed out an error in the featured solution of problem 1128 published in the May 2019 edition. Details of the error as well as an alternate solution will be published in a later issue. We apologize for the error.