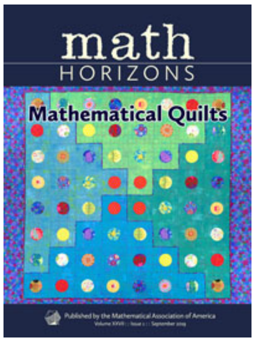

Math Horizons

ISSN: 1072-4117 (Print) 1947-6213 (Online) Journal homepage: https://maa.tandfonline.com/loi/umho20

The Playground

THE PLAYGROUND

Welcome to the Playground! Playground rules are posted on page 33, except for the most important one: *Have fun!*

THE SANDBOX

In this section, we highlight problems that anyone can play with, regardless of mathematical background. But just because these problems are easy to approach doesn't mean that they are easy to solve!

Circle-M Ranch (P390). This problem was submitted by Arsalan Wares (Valdosta State University). The logo for Circle-M Ranch consists of circle m together with five points A, B, C, D, and E arranged counterclockwise on m such that $\angle AEB = \angle EBD = \angle BDC = \tau / 8$ (where $\tau = 2\pi$ is the full angle measure of a circle). Two roughly triangular regions of m are shaded in the logo: one bounded by AE, EB, and acute arc BA, and the other bounded by BD, DC, and acute arc CB, as shown in figure 1. What fraction of the area of m is shaded?

Hyperrational Cheese (P391). This rival of the clever chef in last issue's Carousel problem C25 was created by Hidefumi Katsuura of San Jose State University.

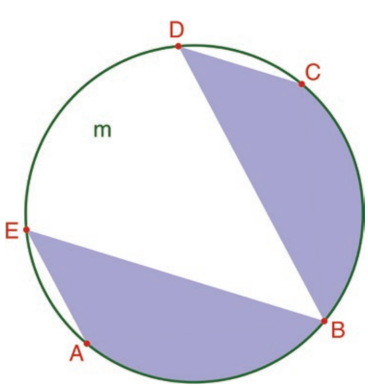

Figure 1. Circle-M Ranch logo.

The chef in problem C25 created two pieces of cheese, one a cylinder of unit radius and depth, and the other a triangular prism. The triangular cross section of the prism is isosceles with base 2 and height 1, and the depth of the prism is 2. The cylinder rests on the xy-plane on its circular face, and the prism lies on its only square face. The top edge of the triangular cheese is parallel to the x-axis (see figure 2). Faced with the irrational ratio of the volumes of these cheeses, the chef slices each one along the plane $z = y$ and discards the larger piece of each to produce two cheeses with identical volume.

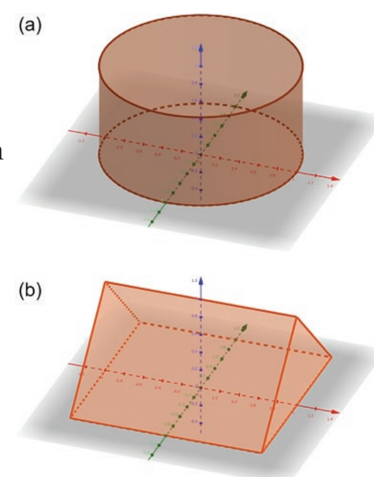

Figure 2. The geometric shapes of two cheeses.

But now the sous-chef comes in and says, "The chef threw out over 70% of the cheese and should be fired for wastefulness! Allow me to demonstrate on this new pair of cheeses." The sous-chef cuts the cylindrical cheese along the planes $z = y$ and $z = -y$, setting aside and keeping both smaller pieces. Then the sous-chef cuts the remaining portion of the cylinder along the planes $z = 1 + x$ and $z = 1 - x$, this time discarding the two smaller pieces and keeping the rest. All of the triangular cheese is kept untouched.

Show that the total volume of the pieces of the cylindrical cheese kept is a rational multiple of the triangular cheese and that the sous-chef kept more cheese than the chef threw away.

THE ZIP-LINE

This section offers problems with connections to articles that appear in the magazine. Not all Zip-Line problems require you to read the corresponding article, but doing so can never hurt, of course.

Fractional Base (P392). This problem lets you Zip-Line to two different recent articles on offbeat number bases by Tom Edgar and coauthors: like this issue's "A Factorial Card Trick" (see page 23), which delves into factoradic numbers; and it is also proposed as a problem in "Happiness is Integral But Not Rational" in the September 2017 issue.

A *sesquinary numeral* is a sequence of digits $d_n d_{n-1} \ldots d_2 d_1 d_0$ where d_n is 1 or 2 and each other d_i is 0, 1, or 2. It represents the (rational) number

$$d_0 + \frac{3}{2} d_1 + \left(\frac{3}{2}\right)^2 d_2 + \cdots + \left(\frac{3}{2}\right)^{n-1} d_{n-1} + \left(\frac{3}{2}\right)^n d_n.$$

Note that a sesquinary numeral might or might not represent a whole number; for example, sesquinary 2101 represents $2 \cdot \frac{27}{8} + \frac{9}{4} + 1 = 10$ while 201 represents $2 \cdot \frac{9}{4} + 1 = \frac{11}{2}$. Show that there is a unique sesquinary numeral representing each positive integer.

THE JUNGLE GYM

Any type of problem may appear in the Jungle Gym—climb on!

Pell Tangent (P393). This problem comes to us from Universidad de Las Palmas de Gran Canaria, Spain, courtesy of Angel Plaza De La Hoz. Let P_k be the kth *Pell number*, that is, the denominator of the kth rational approximation (called a *convergent*) to $\sqrt{2}$ produced by cutting off the infinite repeating continued fraction $\sqrt{2} = [1; \overline{2}]$ after k terms. Thus, $P_3 = 5$ because

$$1 + \cfrac{1}{2 + \cfrac{1}{2}} = \frac{7}{5}$$

has denominator 5. By convention, $P_0 = 0$ and $P_1 = 1$. Compute

$$\sum_{k=1}^{\infty} \arctan\left(\frac{P_k + P_{k-1}}{P_k P_{k+1} - 1}\right) \arctan\left(\frac{P_{k+1} + P_k}{P_k P_{k+1} - 1}\right).$$

FEBRUARY WRAP-UP

Tough Pill to Swallow (P382). Each day, Professor Kibacha takes a half a pill as follows: she dumps one item out of the bottle (uniformly randomly selected). If it's a half pill, she swallows it; otherwise, she breaks it, takes half, and puts the other half back. One day, a whole pill shakes out but drops down the drain. After counting the whole and half pills remaining, the professor thinks, "Hmm, losing that pill did not change the expected number of half pills in the bottle when I end up breaking the last whole pill." What is the difference between the number of whole pills and half pills left in the bottle?

We received the following solution from Randy K. Schwartz (Schoolcraft College): There are **two** more half pills than whole pills after Prof. Kibacha drops one down the drain. Let $E(w,h)$ be the expected number of half pills in the bottle when the last whole pill is broken, given

THE CAROUSEL—OLDIES, BUT GOODIES

In this section, we present an old problem that we like so much, we thought it deserved another go-round. Try this, but be careful—old equipment can be dangerous. Answers appear at the end of the column.

Augmented Pyramid (C26). Take a right regular hexagonal pyramid with base edge length $1/\sqrt{2}$ and height $\sqrt{3}/2$. Adjoin on every other triangular face a triangular pyramid such that the lengths of the three edges that join at the apex are 1, 1/2, and 1/2 (see figure 3). What is the volume of the resulting heptahedral solid?

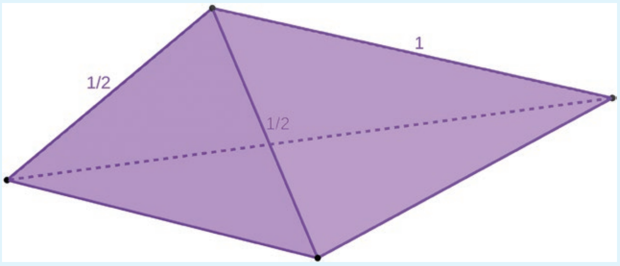

Figure 3. Tetrahedron with which to augment three faces of the pyramid.

that Prof. Kibacha wakes up with w whole pills and h half pills in the bottle. We then have

$$E(1,h) = \frac{1}{h+1}h + \frac{h}{h+1}E(1,h-1)$$

since there is a $1/(h+1)$ chance of choosing the last whole pill, and for $w > 1$,

$$E(w,h) = \frac{w}{w+h}E(w-1,h+1) + \frac{h}{w+h}E(w,h-1).$$

Note that we can subsume the first formula into the second if we simply set (by convention) $E(0,h+1) = h$.

Using these conventional values and the recurrence relation, we can make a table of E for small w and h (like table 1), from which it is not hard to guess the following closed form: $E(w,h) = \frac{h}{w+1} + H_w - 1$, where $H_w = \sum_{i=1}^{w} \frac{1}{i}$ is the wth harmonic number. This form in turn can be proved by induction; if we assume it for all smaller values of w and all smaller values of h, then

$$\begin{aligned} E(w,h) &= \frac{w}{w+h}\left(\frac{h+1}{w} + H_{w-1} - 1\right) \\ &\quad + \frac{h}{w+h}\left(\frac{h-1}{w+1} + H_w - 1\right) \\ &= \frac{h+1}{w+h} - \frac{1}{w+h} + \frac{h}{w+h}\cdot\frac{h-1}{w+1} + H_w - 1 \\ &= \frac{h}{w+1} + H_w - 1. \end{aligned}$$

On the morning in question, suppose there are w whole pills and h half pills in the bottle when counted. We are told $E(w+1,h) = E(w,h)$. Using the formula, this means $\frac{h}{w+2} + H_{w+1} = \frac{h}{w+1} + H_w$. We can rewrite this as $\frac{h}{w+2} = \frac{h-1}{w+1}$, or $h = w+2$, as claimed.

Table 1. Expected remaining half-pills for small values of *w* and *h.*

	h					
w	**0**	**1**	**2**	**3**	**4**	**5**
0		0	1	2	3	4
1	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
2	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{7}{6}$	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{13}{6}$
3	$\frac{5}{6}$	$\frac{13}{12}$	$\frac{4}{3}$	$\frac{19}{12}$	$\frac{11}{6}$	$\frac{25}{12}$

Triangular Triad (P383). This problem was submitted by Arsalan Wares from Valdosta State University. Point K is on side BC of triangle ABC and M and N are the midpoints of the sides adjacent to A, as shown in figure 4. Point P is the intersection of AK and MN, and Q is the intersection of AK with median CM. If triangles APN and MPQ both have area 5 units, what is the area of triangle CKQ?

Triangle CKQ has area 5. We received solutions from Kirtan Jani (North Central College), Jessica Lefler (Slippery Rock University), Randy K. Schwartz (Schoolcraft College), Vasile Teodorovici (NSERC Canada), and problem-solving groups from Cal Poly Pomona and Skidmore College, as well as a partial solution from Dmitry Fleischman. Remarkably, all of the approaches were different, but this method from Vasile proceeds most directly to the goal.

From the givens, $\triangle ANM$ and $\triangle AQM$ have equal area; they also share base AM. Therefore, NQ is parallel to AB; extend this line to meet CB in L as shown. As now $\triangle CAB$ is similar to $\triangle CNL$, L is the midpoint of CB and LM is parallel to CA. Therefore, $CLMN$ is a parallelogram with centroid Q. Hence, $\triangle CKQ \cong \triangle MPQ$, and so has area 5.

Scattered Seed (P384). Looking at the diagrams in Andrew Simoson's article, "A Phyllotaxis of the Irrational," the average number of seeds per unit area goes to zero as the radius r goes to infinity. Similarly, the density in a logarithmic phyllotaxis diagram grows without bound. For a strictly increasing function f, define the *f-ic phyllotaxis diagram* for a real constant $\alpha \in (0,1]$ to be the set of points (in polar coordinates) $\{(f(n), n\alpha\tau \mid n \in \mathbb{Z}^+\}$. Describe the functions f for which the average number of points per unit area in this diagram converges to a positive limit as the radius goes to infinity. Does the value of the limit and/or the collection of such functions f depend on the value of α?

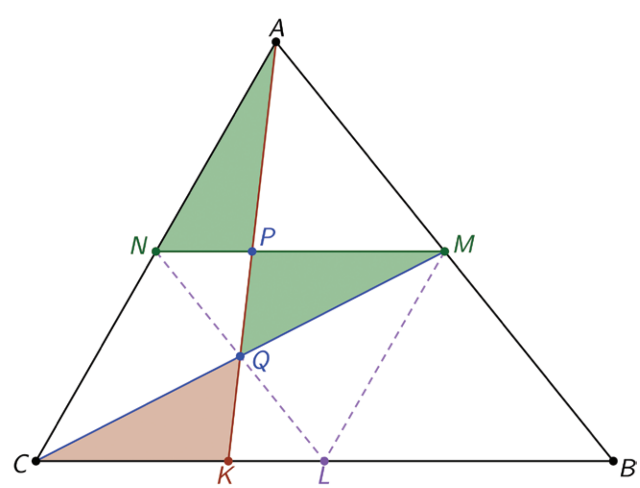

Figure 4. The triad of triangles, with auxiliary lines.

Randy K. Schwartz (Schoolcraft College) submitted a solution. The first N points of the given set lie in a circle of radius $f(N)$, so the average density of the first N points is $\frac{2N}{\tau f(N)^2}$. This ratio must approach a positive limit as $N \to \infty$. As the square root function is continuous and τ is constant, that condition is equivalent to $\sqrt{N}/f(N)$ having a positive limit as $N \to \infty$. From elementary calculus, as $\sqrt{N}$ and $f(N)$ are positive and monotone increasing, this happens if and only if $f(N)/\sqrt{N}$ has a positive limit.

In other words, there is a positive constant k such that for any $\epsilon > 0$, $|k - f(N)/\sqrt{N}| < \epsilon$. This condition happens exactly for the functions $f(N)$ of the form $f(N) = k\sqrt{N} + g(N)$, where g is an arbitrary function such that $g(x)/\sqrt{x} \to 0$ as $x \to \infty$. Note that the value of α affects neither this class of functions nor the limiting average density for a specific such function (that density depends only on k).

Collatz Warmup (P385). Previous "Playground" editor Gary Gordon returns with this problem. For a positive integer n, let $s(n)$ be $n/2$ if n is even and $n+1$ otherwise. Starting from any number, we can apply s repeatedly to obtain a sequence, for example:

$$11 \mapsto 12 \mapsto 6 \mapsto 3 \mapsto 4 \mapsto 2 \mapsto 1.$$

1) Let a_n be the number of steps it takes n to get to 1 (so, $a_{11} = 6$). Find the successive "slowest" values of n; that is, find all numbers n such that $a_n > a_m$ for all $m < n$.

2) Let b_l be the number of times l appears as a_n for some n.

We received solutions from Dmitry Fleischman, Luke Musgrave and Angel Skuse (North Central College), and Randy K. Schwartz (Schoolcraft College), as well as partial solutions from Vasile Teodorovici (NSERC Canada) and Carl Libis (Columbia Southern University). All of the full solutions ran roughly along these lines.

Note that $s(s(2^j+1)) = 2^{j-1}+1$, whence $a_{2^j+1} = 2j+1$. On the other hand, we can see by induction that $2^j + 1 \le n \le 2^{j+1}$ implies $a_n \le 2j+1$: If n is even, then $s(n) = n/2$ is between $2^{j-1}+1$ and 2^j, so $a_{s(n)} \le 2(j-1)+1$ and $a_n = 1 + a_{s(n)} \le 2j < 2j+1$. If n is odd, $s(s(n)) = (n+1)/2$ is between $2^{j-1}+1$ and 2^j, so $a_n \le 2(j-1)+1+2 = 2j+1$. Therefore, the slowest n are exactly the numbers of the form 2^j+1.

For the second part, let $B_n = \{m | a_m = n\}$ so that $b_n = |B_n|$. We note that $|B_{n+1}| = |B_n| + |B_{n-1}|$: For each $m \in B_n$, $2m \in B_{n+1}$, and for each $m \in B_{n-1}$, $2m-1 \in B_{n+1}$. Moreover, if l even is in B_{n+1}, then $m/2 \in B_n$, and if l odd is in B_{n+1}, then $(m+1)/2 \in B_{n-1}$, establishing the claim. Now the fact that $B_1 = \{2\}$ and $B_2 = \{4\}$ are singletons shows that b_n is the nth Fibonacci number.

CAROUSEL SOLUTION

As shown in figure 5, the statement of the problem is simply a complicated way of describing the half of a unit cube created by bisecting it with a plane perpendicular to a body diagonal. Therefore, its volume is 1/2.

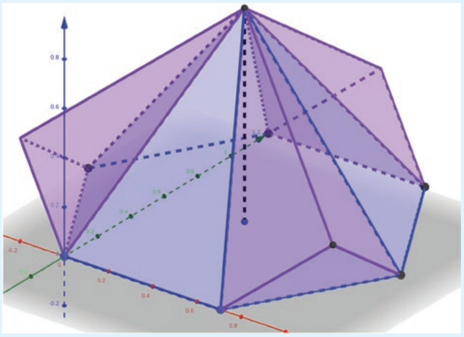

Figure 5. Half a cube dissected as a hexagonal pyramid and three congruent tetrahedra.

CORRECTION

Brian Beasley (Presbyterian College) submitted a solution for Threefold Square (P375) and Jayin Avadov (ADA University) submitted one for Base Tangent Inequality (P377). Their names were regrettably not included when the solutions for these problems first appeared.

SUBMISSION & CONTACT INFORMATION

The Playground features problems for students at the undergraduate and (challenging) high school levels. Problems and solutions should be submitted to *MHproblems@maa.org* and *MHsolutions@maa.org*, respectively (PDF format preferred). Paper submissions can be sent to Glen Whitney, ICERM, 121 South Main Street, Box E, 11th Floor, Providence, RI 02903. Please include your name, email address, and school affiliation, and indicate if you are a student. If a problem has multiple parts, solutions for individual parts will be accepted. Unless otherwise stated, problems have been solved by their proposers.

The deadline for submitting solutions to problems in this issue is November 9.

10.1080/10724117.2019.1629214