Mathematics Magazine

ISSN: 0025-570X (Print) 1930-0980 (Online) Journal homepage: https://maa.tandfonline.com/loi/umma20

Problems and Solutions

PROBLEMS

Proposals

To be considered for publication, solutions should be received by November 1, 2019.

2071. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let $n > 1$ be an integer, and let $\mathbb{Z}_n$ be the ring of integers modulo n. For fixed $k \in \mathbb{Z}_n - \{0\}$, define a binary operation "$\circ$" on $\mathbb{Z}_n$ by $x \circ y = (x-k)(y-k)+k$ for all $x, y \in \mathbb{Z}_n$. Let U be the group of units of $\mathbb{Z}_n$ (under multiplication), and let U_k° be the set of elements of $\mathbb{Z}_n$ that are invertible under the operation $\circ$. Characterize those n with the property that $U \neq U_k^\circ$ for all $k \in \mathbb{Z}_n - \{0\}$.

2072. *Proposed by Julien Sorel, Piatra Neamt, PNI, Romania.*

(a) Show that the initial value problem

$$\begin{cases} y' = \sqrt{1-y^2}, \\ y(0) = 1 \end{cases}$$

has infinitely many solutions defined on $\mathbb{R}$.

(b) By contrast, show that the initial value problem

$$\begin{cases} y' = \sqrt{x^2-y^2}, \\ y(1) = 1 \end{cases}$$

has no solutions defined on an open interval containing $x = 1$.

Math. Mag. **92** (2019) 230–238. doi:10.1080/0025570X.2019.1599245. © Mathematical Association of America

We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at http://mathematicsmagazine.submittable.com. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by* LaTeX *source. General inquiries to the editors should be sent to* mathmagproblems@maa.org

2073. *Proposed by Enrique Treviño, Lake Forest College, Lake Forest, IL.*

A *factorial expansion* is any formal expression of the form

$$\overline{a_k a_{k-1} \dots a_2 a_1},$$

where $a_1, a_2, \dots, a_k$ are k integers ($k \geq 1$) such that $0 \leq a_i \leq i$ for $i = 1, 2, \dots, k$. The *value* of such a factorial expansion is

$$a_k \cdot k! + a_{k-1} \cdot (k-1)! + \cdots + a_2 \cdot 2! + a_1 \cdot 1!.$$

If the integers $a_1, \dots, a_k$ are expressed in base 10 and their digits simply written together without separation, the value of the factorial expansion so written is often ambiguous. For instance, the expansion $\overline{10000000000}$ may be interpreted as having coefficients 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 and value $1 \times 11! + 0 \times (10! + 9! + \cdots + 1!) = 11!$, or having coefficients 10, 0, 0, 0, 0, 0, 0, 0, 0, 0 and value $10 \times 10! + 0 \times (9! + 8! + 7! + \cdots + 1!) = 10 \times 10!$. Such factorial expansions are called *ambiguous*. On the other hand, some factorial expansions are unambiguous: for example, the expansion $\overline{311}$ must have the value $3 \times 3! + 1 \times 2! + 1 \times 1! = 21$. Prove that there are only finitely many unambiguous factorial expansions, and find the one whose value is largest.

2074. *Proposed by Bao Do (student), Columbus State University, Columbus, GA.*

Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} H_k,$$

where $H_k = \sum_{j=1}^{k} \frac{1}{j}$ is the kth harmonic sum.

2075. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD and Mark Kaplan, Towson University, Towson, MD.*

Consider the sequence $\{C_n\}$ defined recursively by $C_0 = 3$, $C_1 = 1$, $C_2 = 3$, and

$$C_n = C_{n-1} + C_{n-2} + C_{n-3} \qquad \text{for } n \geq 3.$$

Let $O = (0, 0, 0)$ be the origin of $\mathbb{R}^3$ and, for integer $n \geq 0$, let P_n be the point (C_n, C_{n+1}, C_{n+2}).

(a) Find the volume of the pyramid $O P_n P_{n+1} P_{n+2}$ in closed form.

(b) Show that the sequence $\{P_n\}$ asymptotically approaches a fixed line $\mathcal{L}$ through the origin of $\mathbb{R}^3$, and characterize this line.

Quickies

1091. *Proposed by H. A. ShahAli, Tehran, Iran.*

Show that no more than two straight cuts are needed to split any triangle into three or fewer pieces that may be rearranged (without overlap or gap) to make a right triangle.

1092. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Given a positive integer n, evaluate

$$\int_0^1 \int_0^1 \arctan\left(\frac{x^n}{y^n}\right) dx\, dy.$$

Solutions

A characterization of S_m as subgroup of S_n for $m < n$ **June 2018**

2046. *Proposed by Ioan Băetu, Botoşani, Romania.*

For integers m, n such that $1 \le m < n$, let S_n be the group of all permutations of $\{1, 2, \ldots, n\}$, let F be the set of permutations $\sigma \in S_n$ such that $\sigma(m) < \sigma(m+1) < \cdots < \sigma(n)$, and let T be the set of transpositions in F. Prove that there exists a unique subgroup G of S_n such that $T \subset G \subset F$.

Solution by Joseph DiMuro, Biola University, La Mirada, CA.
We show that the only subgroup G of S_n satisfying the given hypotheses is $G = S_m$ regarded as the set of permutations in S_n fixing each of $m+1, \ldots, n$.

First, any transposition $\sigma \in S_m$ satisfies $\sigma(m) \le m$ and $\sigma(i) = i$ for all $i > m$, hence $\sigma(m) \le m < m+1 = \sigma(m+1) < \cdots < n = \sigma(n)$, so $\sigma \in T$. Therefore, T contains all transpositions in S_m; thus, G includes the group generated by those transpositions, which is S_m itself.

Conversely, we show that G contains no other permutations. Let $\tau \in S_n - S_m$ be arbitrary. Since $\tau \notin S_m$, τ is not the identity permutation, hence there exists $i \in \{1, \ldots, n\}$ such that $\tau(i) \ne i$. Since $\tau \notin S_m$, the largest such i must satisfy both $i \ge m+1$ and $\tau(k) = k$ for $k = i+1, \ldots, n$. Since τ is injective we cannot have $\tau(i) > i$, hence $\tau(i) < i$. We have $j := \tau^{-1}(i) \ne i$ (since $\tau(i) \ne i$), and so $j < i$ by the choice of i as the largest non-fixed point of τ. Thus, we have $j < i$ but $\tau(j) = i > \tau(i)$. If $j \ge m$, it follows immediately from the definition of F that $\tau \notin F$, and hence $\tau \notin G$ *a fortiori* since $G \subset F$. On the other hand, if $j < m$, let $\sigma = (jm) \in S_m$ be the permutation that transposes j and m. Then we have $\tau\sigma(m) = \tau(j) = i > \tau(i) = \tau\sigma(i)$. Since $m < i$ but $\tau\sigma(m) > \tau\sigma(i)$, we have $\tau\sigma \notin F$ in this case, and so $\tau\sigma \notin G$. Since G is a group and $\sigma \in S_m \subset G$, it follows that $\tau \notin G$ (otherwise we would have $\tau\sigma \in G$). Thus, G contains no permutations $\tau \notin S_m$, so $G = S_m$.

Also solved by Paul Budney, Robert Calcaterra, William Cowieson, Dmitry Fleischman, Neville Fogarty, Abhay Goel, Tom Jager, Peter McPolin (Northern Ireland), Michael Reid, Nikhil Sahoo, and the proposer. There was 1 incomplete or incorrect solution.

A limit-ratio test for convergence to zero **June 2018**

2047. *Proposed by George Stoica, Saint John, New Brunswick, Canada.*

Let (a_n) be a sequence of nonzero real numbers such that

$$\lim_{n\to\infty} n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right)$$

exists and is strictly positive. Prove or disprove: The sequence (a_n) is necessarily convergent.

Solution by Nikhil Sahoo (student), Berkeley City College, Berkeley, CA.
We show that the sequence (a_n) converges to zero under the weaker hypothesis that $L := \liminf_{n\to\infty} n(|a_n/a_{n+1}| - 1)$ is strictly positive.

Lemma. For any positive integers m, n, we have

$$0 < P_{m,n} := \prod_{k=1}^{n-1} \frac{mk}{mk+1} \leq \frac{1}{\sqrt[m]{n}}.$$

(Per the usual convention on empty products, we let $P_{m,1} = 1$.)
Proof. Clearly, $P_{m,n} > 0$. The function $x \mapsto x/(x+1)$ is positive and increasing on $(0, \infty)$; therefore,

$$(P_{m,n})^m = \prod_{j=0}^{m-1} P_{m,n} = \prod_{k=1}^{n-1}\prod_{j=0}^{m-1} \frac{mk}{mk+1} \leq \prod_{k=1}^{n-1}\prod_{j=0}^{m-1} \frac{mk+j}{mk+j+1} = \prod_{i=m}^{mn-1} \frac{i}{i+1} = \frac{m}{mn}$$
$$= \frac{1}{n}.$$

Taking m-roots of both sides of the inequality above concludes the proof of the lemma.

For fixed M, it follows from the lemma that $\lim_{n\to\infty} P_{M,n} = 0$ since $1/\sqrt[M]{n} \to 0$ as $n \to \infty$. By the assumption that the limit in the statement of the problem is positive, there exist positive integers M and N such that $n \geq N$ implies $n(|a_n/a_{n+1}| - 1) \geq 1/M$; equivalently,

$$|a_{n+1}| \leq \frac{Mn}{Mn+1} \cdot |a_n| \qquad \text{for all } n \geq N.$$

It follows by induction that $|a_n| \leq |a_N| \cdot \prod_{k=N}^{n-1} \left[Mk/(Mk+1)\right] = |a_N| \cdot P_{M,n}/P_{M,N}$ for $n \geq N$. Since $\lim_{n\to\infty} P_{M,n} = 0$, we see that (a_n) converges to zero.

Editor's Note. Christopher Hammond remarked that the value of the limit $L = \lim_{n\to\infty} n(|a_n/a_{n+1}| - 1)$ is closely related to Raabe's test for convergence of the series $\sum_{n=1}^{\infty} a_n$. Christopher N. B. Hammond, The Case for Raabe's Test, *Mathematics Magazine* (forthcoming). The series converges absolutely when $L > 1$ and diverges when $L < 0$. For $0 \leq L < 1$, the series may either diverge or converge conditionally (Theorem 1 therein). However, if $L > 0$ then the sequence (a_n) necessarily converges to zero as asserted in the statement of the problem (Proposition 2 therein).

Also solved by Ulrich Abel & Vitaliy Kushnirevych (Germany), Robert A. Agnew, Paul Budney, William Cowieson, Souvik Dey, Joseph DiMuro, Robert L. Doucette, Dmitry Fleischman, Abhay Goel, Russell Gordon, Christopher N. B. Hammond, Lixing Han, Eugene A. Herman, Tom Jager, John C. Kieffer, Jimin Kim (South Korea), Elias Lampakis (Greece), Kee-Wai Lau, Peter McPolin (Northern Ireland), Albert Natian, Northwestern University Math Problem Solving Group, Moubinool Omarjee (France), Michael Reid, Celia Schacht, Christopher Sinkule, Nora Thornber, Lawrence R. Weill, and the proposer. There were 2 incomplete or incorrect solutions.

A random triangle with vertices in a three-quarter disk **June 2018**

2048. *Proposed by Julien Sorel, Piatra Neamt, PNI, Romania.*

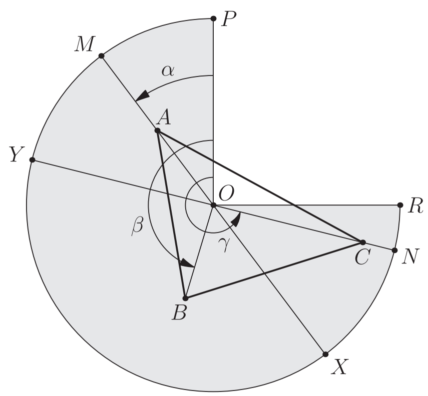
Three points A, B, C are chosen uniformly at random in the three-quarter disk

$$\mathcal{Q} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1, \text{ and either } x \leq 0 \text{ or } y \leq 0\}$$

obtained by removing the first quadrant of the unit disk. What is the probability that the origin $O = (0, 0)$ lies inside $\triangle ABC$?

Solution by Xueshi Gao (student), Peking University, Beijing, China.
We prove that the event $\mathcal{E}$ that O lies inside $\triangle ABC$ has probability $\mathbf{P}[\mathcal{E}] = 5/27$.

Let $P = (0, 1)$, $R = (1, 0)$, and $\alpha = \angle POA$, $\beta = \angle POB$, $\gamma = \angle POC$, so that $\alpha, \beta, \gamma \in [0, 3\pi/4]$. Let $\overline{MX}$ be the diameter through A so $AM < AX$, and $\overline{NY}$ the diameter through C so $CN < CY$. Consider the event $\mathcal{E}' = \mathcal{E} \cap \{\alpha < \beta < \gamma\}$ as shown in the figure below.



By the assumption that O lies inside $\triangle ABC$, each of the three angles $\angle COA$, $\angle AOB$ and $\angle BOC$ must be strictly less than a half revolution. It follows that point A must lie in the second quadrant, C in sector ROX, and B in sector XOY, that is,

$$0 \leq \alpha < \frac{\pi}{2}, \quad \alpha + \pi < \gamma \leq \frac{3}{2}\pi, \quad \text{and} \quad \gamma - \pi < \beta < \alpha + \pi.$$

Observe that angles α, β, γ are independent and uniformly distributed in $[0, 3\pi/2]$ because A, B, C are independent and uniformly distributed in $\mathcal{Q}$; therefore, the probability of event $\mathcal{E}'$ is

$$\mathbf{P}\left[\mathcal{E}'\right] = \int_0^{\frac{\pi}{2}} \int_{\alpha+\pi}^{\frac{3}{2}\pi} \int_{\gamma-\pi}^{\alpha+\pi} \left(\frac{2}{3\pi}\right)^3 d\beta\, d\gamma\, d\alpha = \frac{5}{162}.$$

By independence of the three random points A, B, C and the fact that event $\mathcal{E}$ is invariant under permutations thereof, we have

$$\mathbf{P}\left[\mathcal{E}\right] = \frac{\mathbf{P}\left[\mathcal{E}'\right]}{\mathbf{P}[\{\alpha < \beta < \gamma\}]} = \frac{5/162}{1/6} = \frac{5}{27}.$$

Also solved by Herb Bailey & Mark Bailey, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Bill Cowieson, Bao Do, Gregory Dresden, John N. Fitch, Neville Fogarty, Kyle Gatesman, GWstat Problem Solving Group, Elias Lampakis (Greece), Albert Natian, Mingyu Park (Korea), Sung Hee Park (Korea), Nikhil Sahoo, Jacob Siehler, Nora S. Thornber, Lawrence Weill, and the proposer. There were 2 incomplete or incorrect solutions.

Non-sparse disjoint subsets of any finite collection of squares **June 2018**

2049. *Proposed by Scott Duke Kominers, Harvard University, Cambridge, MA.*

Show that any finite set of squares in the plane (possibly of different sizes and not necessarily disjoint) has a subset consisting of non-overlapping squares that together cover at least 7% of the area covered by the full set.

Solution by Jimin Kim (student), Institute of Science Education for the Gifted and Talented, Yonsei University, Republic of Korea.
Let $\mathcal{C}$ be the given finite collection of squares. The assertion is trivial when $\mathcal{C}$ is empty, so we assume $\mathcal{C}$ is nonempty henceforth. Successively choose squares $S_1, S_2, \ldots$ in $\mathcal{C}$ by the following recursive method:

- Let S_1 be any square of largest area in $\mathcal{C}$.
- Having chosen $S_1, \ldots, S_i$, let S_{i+1} be any square in $\mathcal{C}$ having largest area among those squares in $\mathcal{C}$ that intersect neither of $S_1, \ldots, S_i$. If there is no such square, the procedure terminates.

The procedure must terminate after choosing a finite number $m \geq 1$ of squares $S_1, S_2, \ldots, S_m$ since $\mathcal{C}$ is finite by hypothesis; because of this, every square in $\mathcal{C}$ intersects some S_i. (Were there any squares in $\mathcal{C}$ intersecting no S_i, any such of largest area would allow the recursive procedure to continue!) Thus, we have $\mathcal{C} = \bigcup_{i=1}^{m} \mathcal{N}_i$ where $\mathcal{N}_i$ $(i = 1, \ldots, m)$ denotes the set of squares in $\mathcal{C}$ that intersect S_i but no S_j with $j < i$. Certainly, we have $S_i \in \mathcal{N}_i$ (since S_i intersects itself, but is otherwise chosen not to intersect S_j for any $j < i$). The sets $\mathcal{N}_1, \ldots, \mathcal{N}_m$ are clearly disjoint by construction; in particular, the squares $S_1, \ldots, S_m$ are disjoint. Moreover, S_i by choice has largest area among all squares in $\mathcal{N}_i \cup \cdots \cup \mathcal{N}_m$ (which is the set of squares in $\mathcal{C}$ intersecting none of $S_1, \ldots, S_{i-1}$); in particular, S_i has largest area among all squares in $\mathcal{N}_i$. We will show that the set consisting of the disjoint squares $S_1, \ldots, S_m$ covers at least 7% of the area covered by all squares in $\mathcal{C}$.

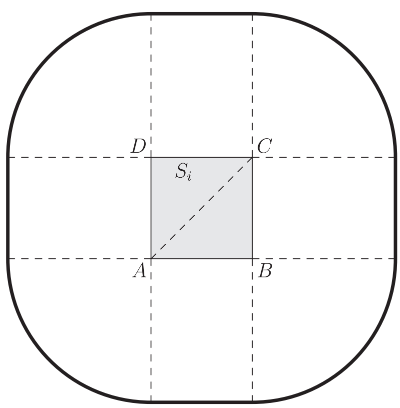


Figure 1 The region R_i consisting of points lying at distance no more than d from square $S_i = \square ABCD$. All dashed segments have the same length $d = AC$.

Let S_i be a square $\square ABCD$ with side $\ell = AB$ and diagonal $d = \sqrt{2}\ell = AC$. Any square $Q \in \mathcal{N}_i$ intersects S_i and has area, hence also diagonal length, not exceeding that of S_i. It follows that Q is fully covered by the region R_i (depicted in Figure 1 above) consisting of points at distance no more than d from S_i. Thus, the region $\overline{\mathcal{N}}_i$ covered by all the squares in $\mathcal{N}_i$ is fully covered by R_i, and its area $\left|\overline{\mathcal{N}}_i\right|$ satisfies

$$\left|\overline{\mathcal{N}}_i\right| \leq |R_i| = (2\pi + 4\sqrt{2} + 1)|S_i|.$$

(We denote by $\left|\mathcal{X}\right|$ the area of a region $\mathcal{X}$ of the plane.) The collection $\mathcal{C}$ covers the region $\overline{\mathcal{C}} = \overline{\mathcal{N}}_1 \cup \cdots \cup \overline{\mathcal{N}}_n$, so we have

$$\left|\overline{\mathcal{C}}\right| = \left|\bigcup_{i=1}^{m} \overline{\mathcal{N}_i}\right| \leq \sum_{i=1}^{m} \left|\overline{\mathcal{N}}_i\right| \leq (2\pi + 4\sqrt{2} + 1) \sum_{i=1}^{m} |S_i|.$$

Therefore, the disjoint squares $S_1, \ldots, S_m$ cover an area

$$\sum_{i=1}^{m} |S_i| \geq \frac{1}{2\pi + 4\sqrt{2} + 1} \left|\overline{\mathcal{C}}\right| = 0.077\cdots \times \left|\overline{\mathcal{C}}\right|$$
$$> 0.07 \times \left|\overline{\mathcal{C}}\right|.$$

Also solved by Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Kyle Gatesman, Abhay Goel, Sarah Kapinos & J. Todd Lee, Northwestern University Math Problem Solving Group, Nikhil Sahoo, Celia Schacht, Lawrence R. Weill, and the proposer.

Counting de Bruijn sequences of pairs of three symbols **June 2018**

2050. *Proposed by Sung Soo Kim, Hanyang University, Ansan, Korea.*

Find the number of sequences $a_1, a_2, \ldots, a_9$ in $\{1, 2, 3\}$ such that

(*i*) $a_1 = a_2 = 1$, and
(*ii*) the nine pairs $(a_1, a_2), (a_2, a_3), \ldots, (a_8, a_9), (a_9, a_1)$ are the same as the nine pairs $(1, 1), (1, 2), \ldots, (3, 2), (3, 3)$ in some order.

Solution by Skidmore College Problem Group, Saratoga Springs, NY.

Figure 2 Graph A (left) and graph B (right).

We show that there are 24 such sequences. This problem amounts to finding the number of cycles traversing each edge (i.e., Eulerian cycles) in the directed graph A on the left of Figure 2 above, beginning with the loop $(1, 1)$: The sequences $a_1 = 1, a_2 = 1, a_3, \dots, a_9$ of vertices (excluding the final return vertex $a_{10} = 1 = a_1$) in such a cycle are precisely those considered in the problem. We call these "*a-cycles*." There is a four-to-one correspondence between the set of a-cycles and the set of Eulerian cycles successively visiting vertices $1 = b_1, b_2, \dots, b_6$ of graph B on the right of Figure 2 above, starting at vertex 1 (and eventually traversing all edges, returning to vertex $b_7 = 1 = b_1$). We refer to the latter as "*b-cycles*." Indeed, given an a-cycle α we may simply remove the loops $(1, 1)$, $(2, 2)$, $(3, 3)$ from α to obtain a b-cycle β; conversely, given a b-cycle β, simply add the loop $(1, 1)$ at the beginning of β, insert the loop $(2, 2)$ at either of the two occasions when vertex 2 is visited, and the loop $(3, 3)$ at either of the two occasions when vertex 3 is visited—this gives four different a-cycles $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ the removal of whose loops results in β. We proceed to count the number of distinct b-cycles.

With graph B as depicted, we call directed edges $(1, 2)$, $(2, 3)$, and $(3, 1)$ *inner*, and edges $(2, 1)$, $(3, 2)$, and $(1, 3)$ *outer*. A moment's reflection shows that, given a vertex b_i of any path on graph B, the next vertex b_{i+1} is uniquely determined by specifying whether b_{i+1} is reached from b_i by traversing an inner or an outer edge. The set of b-cycles is partitioned into the disjoint sets of *inner* b-cycles that start with the inner edge $b_1 = 1 \to 2 = b_2$, and the set of *outer* b-cycles that start with the outer edge $b_1 = 1 \to 3 = b_2$. Given the identical role played by vertices 2 and 3 in the problem, it is clear that there are equally many inner and outer b-cycles. (Exchanging the labels of vertices 2 and 3 is a bijection between the sets of inner and outer b-cycles.) Thus, it suffices to count inner b-cycles. These are easily seen to be encoded by the sequences

iooii, iioooi, iiiooo

(where "i" means "follow inner edge," while "o" means "follow outer edge"), corresponding to the b-cycles:

$$1 \to 2 \to 1 \to 3 \to 2 \to 3(\to 1),$$

$$1 \to 2 \to 3 \to 2 \to 1 \to 3(\to 1),$$

$$1 \to 2 \to 3 \to 1 \to 3 \to 2(\to 1).$$

Thus, there are 3 distinct inner b-cycles, while the number of b-cycles is $2 \cdot 3 = 6$, and the number of a-cycles (hence of sequences solving the problem) is $4 \cdot 6 = 24$.

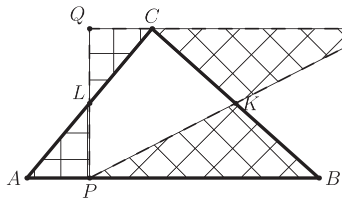
Editor's Note. Rob Pratt remarked that the problem asks for the number of de Bruijn sequences of order $n = 2$ over an alphabet of $k = 3$ symbols, of which there are $k!^{k^{n-1}}/k^n$ in general, and in particular $3!^{3^{2-1}}/3^2 = 24$ in this problem. (T. van Aardenne-Ehrenfest and N. G. de Bruijn, Circuits and trees in oriented linear graphs. *Simon Stevin* **28** (1951) 203–217.) Jacob Siehler brought to our attention that the twenty-four sequences and the general formula above appeared in this MAGAZINE. (Anthony Ralston, De Bruijn sequences—A model example of the interaction of discrete mathematics and computer science, *Mathematics Magazine* **55** (1982) 131–143.)

Also solved by Skyler Addy & Zachary Parker, Brian D. Beasley, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, Timothy Crane, Dmitry Fleischman, Neville Fogarty, Abhay Goel, Eugene A. Herman, Dain Kim (Korea), Brad Meyer, Ioana Mihăilă, North Carolina Wesleyan College Fall 2018 MAT 318 Discrete Methods Class, Rob Pratt, Nikhil Sahoo, Joel Schlosberg, Jacob Siehler, David Stone & John Hawkins, Lawrence R. Weill, and the proposer. There was 1 incomplete or incorrect solution.

Answers

Solutions to the Quickies from pages 231 and 232.

A1091. Label the vertices of the given triangle $\triangle ABC$ so $\angle C$ is its largest angle; thus, $\angle A$ and $\angle B$ are both acute. Let K be the midpoint of $\overline{BC}$ and L the midpoint of $\overline{AC}$. Choose P on $\overline{AB}$ so $\overline{LP}$ is perpendicular to $\overline{AB}$. Note that P lies on side $\overline{AB}$ since $\angle A$ and $\angle B$ are both acute. Cut triangle $\triangle ABC$ along segments $\overline{KP}$ and $\overline{LP}$ thus splitting it into triangles $\triangle ALP$, $\triangle BKP$ and quadrilateral $CKPL$. Keeping the quadrilateral fixed, rotate the right triangle $\triangle ALP$ half a revolution about L to obtain a new triangle $\triangle CLQ$, and triangle $\triangle BKP$ half a revolution about K to obtain triangle $\triangle CKR$. In doing so we obtain a triangle $\triangle PQR$ with angle $\angle Q$ right. (This construction works even if the original triangle was a right triangle to begin.)



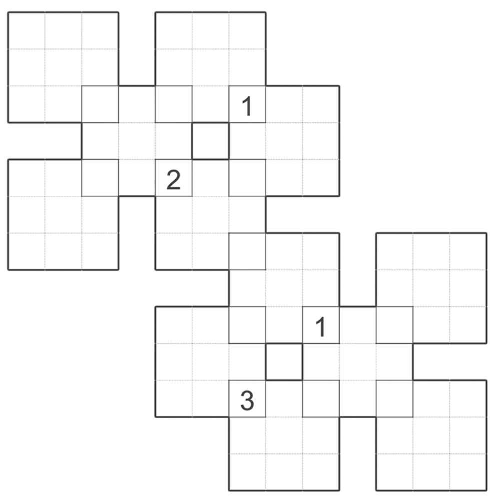
A1092. We show that the double integral has the value $\pi/4$ for all n. Swapping the order of integration and the names of the variables x, y, we have

$$I = \int_0^1 \int_0^1 \arctan\left(\frac{x^n}{y^n}\right) dx\, dy = \int_0^1 \int_0^1 \arctan\left(\frac{y^n}{x^n}\right) dx\, dy.$$

From the identity $\arctan(t) + \arctan(1/t) = \pi/2$ (valid for $t > 0$), we obtain

$$I = \frac{1}{2}\int_0^1 \int_0^1 \left[\arctan\left(\frac{x^n}{y^n}\right) + \arctan\left(\frac{y^n}{x^n}\right)\right] dx\, dy = \frac{1}{2}\int_0^1 \int_0^1 \frac{\pi}{2}\, dx\, dy = \frac{\pi}{4}.$$

Tribus Puzzle



How to play. Fill each of the three-by-three squares with either a 1, 2, or 3 so that each number appears exactly once in each column and row. Some cells apply to more than one square, as the squares overlap. Each of the three-by-three squares must be distinct. The solution can be found in page 172.

— David Nacin, William Paterson University, Wayne, NJ (nacind@wpunj.edu)