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Problems and Solutions

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PROBLEMS

Problem 2067 Updated *Editor's Note.* The statement of Problem 2067 that appeared in the April 2019 issue omitted the critical hypothesis that chord $\overline{MN}$ goes through P. We sincerely regret the mistake, and thank Robert Calcaterra for bringing it to our attention. The corrected statement of Problem 2067 appears below.

2067. *Proposed by Elton Bojaxhiu, Eppstein am Taunus, Germany and Enkel Hysnelaj, Sydney, Australia.*

Chord $\overline{XY}$ of a circle $\mathcal{C}$ is not a diameter. Let P, Q be two different points strictly inside $\overline{XY}$ such that Q lies between P and X. Chord $\overline{MN}$ through P is perpendicular to the diameter of $\mathcal{C}$ through Q, where $MP < NP$. Prove that $(MQ - PQ) \cdot XY \geq 2 \cdot QX \cdot PY$, and characterize those cases in which equality holds.

Proposals

To be considered for publication, solutions should be received by March 1, 2020.

2076. *Proposed by Michael Goldenberg, The Ingenuity Project, Baltimore Polytechnic Institute, Baltimore, MD and Mark Kaplan, Towson University, Towson, MD.*

Given real numbers C_0, C_1, and C_2, one defines a *general Tribonacci (GT) sequence* $\{C_n\}$ recursively by the relation $C_{n+3} = C_{n+2} + C_{n+1} + C_n$ for all $n \geq 0$. Such GT-sequence $\{C_n\}$ is *nonsingular* if

$$\Delta = \begin{vmatrix} C_0 & C_1 & C_2 \\ C_1 & C_2 & C_3 \\ C_2 & C_3 & C_4 \end{vmatrix} \neq 0.$$

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We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this Magazine.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by LaTeX source. General inquiries to the editors should be sent to* mathmagproblems@maa.org.

A *dual Tribonacci (DT) sequence* $\{D_n\}$ is one that satisfies the dual recurrence $D_{n+3} + D_{n+2} + D_{n+1} = D_n$ for $n \geq 0$. Show that for any nonsingular GT-sequence $\{C_n\}$ with C_0, C_1, C_2 positive there exists a DT-sequence $\{D_n\}$ such that, for all $n \geq 0$,

$$\arctan\left(\frac{\sqrt{D_n}}{C_n}\right) = \arctan\left(\frac{\sqrt{D_n}}{C_{n+1}}\right) + \arctan\left(\frac{\sqrt{D_n}}{C_{n+2}}\right) + \arctan\left(\frac{\sqrt{D_n}}{C_{n+3}}\right).$$

2077. *Proposed by Li Zhou, Polk State College, Winter Haven, FL.*

Prove that in any triangle with side lengths a, b, c, inradius r, and circumradius R, we have

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{r}{R} > \frac{5}{3}.$$

2078. *Proposed by Florin Stanescu, Serban Cioculescu School, Gaesti, Romania.*

Let A, B be $n \times n$ complex matrices such that $A^2 + B^2 = 2AB$. Prove that $(AB - BA)^m = \mathbf{0}$ for some $m \leq \lceil \frac{n}{2} \rceil$.

2079. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Given real numbers a, b, with $b > 0$, prove that the integral

$$J(a, b) := \int_0^\infty \left[2 + (x + a)\ln\left(\frac{x}{x+b}\right)\right] dx$$

converges if and only if $a = 1$ and $b = 2$, and find the value $J(1, 2)$.

2080. *Proposed by the UTSA Problem Solving Club, University of Texas at San Antonio, San Antonio, TX.*

For $n \geq 3$, let W_n be the wheel graph consisting of an n-cycle all whose vertices are joined to an additional distinct vertex.

(i) How many colorings of the $2n$ edges of W_n using $k \geq 2$ colors result in no monochromatic triangles?

(ii) Regard two colorings of W_n as equivalent if there is a graph automorphism of W_n that maps the first coloring to the second. If $k \geq 2$ and $p > 3$ is prime, count all non-equivalent colorings of W_p using k colors.

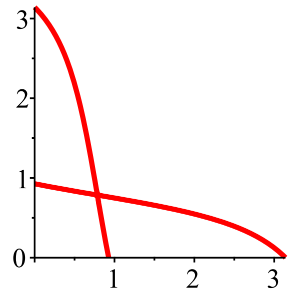
Quickies

1093. *Proposed by Mihaly Bencze, Brasov, Romania.*

Show that 2019^{2n} can be expressed as a sum of ten different positive squares, for every positive integer n.

1094. *Proposed by Julien Sorel, Piatra Neamt, PNI, Romania.*

The curve $2\sin(x + y) - \cos(x - y) = 1$ has a self-intersection point at $(\pi/4, \pi/4)$ as shown in the figure below. Find the angle between the two tangent lines to the curve at this point.



Solutions

The largest roots of a sequence of polynomials **October 2018**

2051. *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.*

For any positive integer n consider the polynomial $P_n(x) = x^4 - nx^3 - nx^2 - nx + 1$ and let a_n be the largest of its real roots. Find

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n^2}.$$

Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
We show that the limit exists and equals $1/2$. If $x \geq n+1$, then

$$\begin{aligned} P_n(x) &= (x-n)x^3 - nx^2 - nx + 1 \geq 1x^3 - nx^2 - nx + 1 \\ &= (x-n)x^2 - nx + 1 \geq 1x^2 - nx + 1 = (x-n)x + 1 \\ &\geq 1x + 1 \geq n + 2 > 0. \end{aligned}$$

On the other hand, $P_n(n) = -n^3 - n^2 + 1 < 0$. By continuity of P_n and the intermediate value theorem, it follows that $n < a_n < n+1$; hence,

$$\frac{n^2+n}{2n^2} = \frac{1}{n^2}\sum_{i=1}^{n} i < \frac{1}{n^2}\sum_{i=1}^{n} a_i < \frac{1}{n^2}\sum_{i=1}^{n}(i+1) = \frac{n^2+3n}{2n^2}.$$

The first and last expressions above have the same limit $1/2$ as n tends to infinity. By the sandwich theorem,

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} a_i = \frac{1}{2}.$$

Also solved by Ulrich Abel (Germany), Terrance Alvarez & Cyane Gonzalez, Michael A. Ask, Michel Bataille (France), Necdet Batir (Turkey), Brian D. Beasley, Anthony J. Bevelacqua, Robert Calcaterra, Robin Chapman (UK), Jyoti Champanerkar, John Christopher, Michael P. Cohen, Bill Cowieson, Antonella Cupillari, Richard Daquila, Robert L. Doucette, Dmitry Fleischman, Charles Fleming, Natacha Fontes-Merz, Michael Goldenberg & Mark Kaplan, Abhay Goel, Dean Gooch, Lixing Han, Kyle Hansen, GWstat Problem Solving Group, Eugene A. Herman, Theo Koupelis, Elias Lampakis (Greece), Jeffery M. Lewis, James Magliano, Peter McPolin (Northern Ireland), Northwestern University Math Problem Solving Group, Michael Reid, Volkhard Schindler, Joel Schlosberg, Edward Schmeichel, Mark Schultz, Randy K. Schwartz, Achilleas Sinefakopoulos (Greece), Nicholas C. Singer, Albert Stadler (Switzerland), David Stone & John Hawkins, Koopa

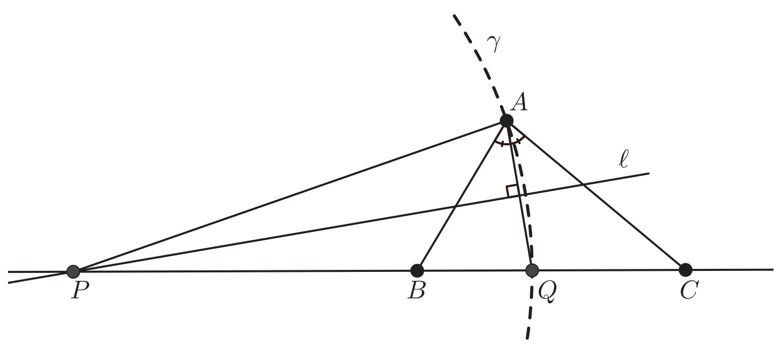
Tak Lun Koo (Hong Kong), The Iowa State Undergraduate Problem Solving Group, Michael Vowe (Switzerland), John Zacharias, and the proposer. There was one incomplete or incorrect solution.

A pencil of lines obtained from any scalene triangle **October 2018**

2052. *Proposed by Michel Bataille, Rouen, France.*

Let $\triangle ABC$ be a scalene triangle. Let D be a variable point on line $\overleftrightarrow{BC}$ such that $D \neq B$ and $D \neq C$. Let E lie on $\overleftrightarrow{BC}$ so $\overleftrightarrow{AE}$ is the reflection of $\overleftrightarrow{AD}$ across the bisector of angle $\angle BAC$. Let O_1, O_2 be the circumcenters of triangles $\triangle ABD$ and $\triangle ACE$, respectively. Prove that there exists a point P, independent of the choice of D, such that line $\overleftrightarrow{O_1O_2}$ passes through P.

Solution by Peter McPolin, St. Mary's University College, Northern Ireland.
Let the bisector of angle $\angle BAC$ intersect $\overline{BC}$ at Q. Since $\triangle ABC$ is not isosceles, the perpendicular bisector ℓ of the segment $\overline{AQ}$ is not parallel to $\overleftrightarrow{BC}$. The point P of intersection of ℓ and $\overleftrightarrow{BC}$ depends only on the choice of the triangle $\triangle ABC$. We show that O_1, O_2, and P are collinear. Let γ be the circle with centre P passing through A (hence also through Q). We prove that the points B and C are inverses with respect to γ.



In the figure above (where $AB < AC$, which may be assumed to hold without loss of generality), we have $\angle PAB + \angle BAQ = \angle PAQ = \angle AQP$ ($\triangle PAQ$ is isosceles with $\overline{PA} = \overline{PQ}$), $\angle BAQ = \angle QAC$ ($\overline{AQ}$ is the bisector of angle $\angle BAC$), $\angle AQP = \angle QAC + \angle ACQ$ ($\angle AQP$ is an exterior angle of triangle $\triangle QAC$), and so $\angle PAB = \angle ACQ = \angle ACP$. Thus, triangles $\triangle PAB$ and $\triangle PCA$ are similar; hence, $PB/PA = PA/PC$, so $PB \cdot PC = PA^2$, showing that B and C are inverses with respect to γ.

If D lies on the half-line $\overrightarrow{PQ}$ then, by construction of E from D, the same line $\overleftrightarrow{AQ}$ bisects angle $\angle EAD$, so the argument above proves that D and E are inverses with respect to γ. If D lies on the other half-line $\overrightarrow{PQ'}$, where Q' is the diametrical opposite of Q on γ, the same conclusion follows upon replacing Q by Q' in the preceding argument. (If $D = P$ then the reflection of $\overleftrightarrow{AD}$ on $\overleftrightarrow{AQ}$ is parallel to $\overleftrightarrow{BC}$, so E is undefined—it may be conventionally regarded as the point at infinity, inverse with respect to γ of its center P.) Inversion with respect to γ fixes A, so this inversion transforms the circumcircle δ of triangle $\triangle ABD$ into the circumcircle ε of triangle $\triangle ACE$. Evidently, the circles γ, δ, ε are coaxial, so their centers P, O_1, and O_2 are collinear; moreover, as indicated above, P is independent of the choice of D.

Also solved by Andrea Fanchini (Italy), Elias Lampakis (Greece), José H. Nieto (Venezuela), Achilleas Sinefakopoulos (Greece), Elton Bojaxhiu (Albania) & Enkel Hysnelaj (Australia) Peter

McPolin (Northern Ireland), Lienhard Wimmer (Germany), Theo Koupelis, Kyle Gatesman and the proposer. There was one incomplete or incorrect solution.

Maximally deranged permutations **October 2018**

2053. *Proposed by Sung Soo Kim, Hanyang University, Korea.*

Let $a = (a_1, a_2, \dots, a_{2018})$ be a permutation of the integers $1, 2, \dots, 2018$. For any integer k in the range $1 \le k \le 2018$, let $l_k(a)$ be the length of the longest monotone subsequence of $(a_k, a_{k+1}, \dots, a_{2018})$ whose first term is a_k, and let $L(a) = \sum_{k=1}^{2018} l_k(a)$. Find the minimum value of $L(a)$ as a ranges over all permutations of $1, 2, \dots, 2018$.

Solution by Michael Reid, University of Central Florida, Orlando, FL.

The minimum value is $\sum_{k=1}^{2018} \left\lceil \sqrt{k} \right\rceil = 61\,440$. We need the following well-known result from combinatorics.

Theorem [P. Erdős, G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica,* **2** (1935) 463–470, https://eudml.org/doc/88611]

Let r, s be natural numbers. A sequence of distinct real numbers having length $> rs$ has either an increasing subsequence of length $> r$, or a decreasing subsequence of length $> s$.

Proof. For a sequence $a = (a_1, a_2, \dots, a_n)$ with $n > rs$, define $f, g : \{1, \dots, n\} \to \mathbb{N}$ as follows: $f(i)$ (resp., $g(i)$) is the length of the longest increasing (resp., decreasing) subsequence of $(a_i, \dots, a_n)$ with first term a_i. The pairs $(f(i), g(i))$ as i varies are all distinct: If $i < j$ and $a_i < a_j$ (resp., $a_i > a_j$), then $f(i) > f(j)$ (resp., $g(i) > g(j)$). By the pigeonhole principle, since $n > rs$, the pairs $(f(i), g(i))$ cannot all lie in $\{1, \dots, r\} \times \{1, 2, \dots, s\}$; thus, either $f(i) > r$ or $g(i) > s$ for some i, whence the conclusion of the theorem follows immediately.

Resuming the solution, for $n \in \mathbb{N}$ and any sequence $a = (a_1, a_2, \dots, a_n)$ of distinct real numbers, define $l_k(a)$ as the length of the longest monotone subsequence of $(a_k, a_{k+1}, \dots, a_n)$ whose first term is a_k, and $L(a)$ as $\sum_{k=1}^{n} l_k(a)$. By induction on n, we will show that

$$L(a) \ge M(n) := \sum_{k=1}^{n} \left\lceil \sqrt{k} \right\rceil \tag{$*$}$$

for every such sequence a of length n. For $n = 1$, inequality $(*)$ holds since both its sides are equal to 1. Next, suppose inequality $(*)$ holds for all sequences of some fixed length n, and let a be a sequence of length $n + 1$. For $r = s = \left\lceil \sqrt{n+1} \right\rceil - 1$, the sequence a has length $n + 1 \ge rs + 1 > r^2$, so its longest monotone subsequence has length (at least) $r + 1 = \left\lceil \sqrt{n+1} \right\rceil$. Let a_i be the first term of a longest such subsequence, so $l_i(a) \ge \left\lceil \sqrt{n+1} \right\rceil$, and let $\hat{a} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{n+1})$ be the length-n sequence obtained from a by deleting the term a_i. For $i < k \le n + 1$, we have $l_k(a) = l_{k-1}(\hat{a})$ ($=$ length of the longest monotone subsequence of $(a_k, a_{k+1}, \dots, a_{n+1})$ whose first term is a_k). For $1 \le k < i$, we have $l_k(a) \ge l_k(\hat{a})$ because any monotone subsequence of $\hat{a}$ starting at a_k is also a monotone subsequence of a starting at a_k.

It follows that

$$
\begin{aligned}
L(a) = l_i(a) + \sum_{k=1}^{i-1} l_k(a) + \sum_{k=i+1}^{n+1} l_k(a) &\geq \left\lceil \sqrt{n+1} \right\rceil + \sum_{k=1}^{i-1} l_k(\hat{a}) + \sum_{k=i+1}^{n+1} l_{k-1}(\hat{a}) \\
&= \left\lceil \sqrt{n+1} \right\rceil + \sum_{k=1}^{n} l_k(\hat{a}) = \left\lceil \sqrt{n+1} \right\rceil + L(\hat{a}) \\
&\geq \left\lceil \sqrt{n+1} \right\rceil + M(n) = M(n+1),
\end{aligned}
$$

by the assumed validity of $(*)$ for the length-n sequence $\hat{a}$. This completes the inductive proof of $(*)$ for all $n \geq 1$.

Call a sequence $a = (a_1, a_2, \ldots, a_n)$ of n distinct numbers *deranged* if $l_k(a) \leq \left\lceil \sqrt{n+1-k} \right\rceil$ for $1 \leq k \leq n$. A deranged sequence satisfies the inequality $L(a) \leq \sum_{k=1}^{n} \left\lceil \sqrt{n+1-k} \right\rceil = M(n)$. By inequality $(*)$, a deranged sequence actually satisfies that $L(a) = M(n)$ is minimum among all sequences of length n.

First, we construct deranged sequences whose length n is an arbitrary perfect square. Any sequence of length $1^2 = 1$ is deranged. Assume a deranged sequence a of length $n = t^2$ has been constructed; we proceed to construct a deranged sequence $\hat{a}$ of length $N = (t+1)^2$. For any choice of $b_1, b_2, \ldots, b_t, b_{t+1}$ and $c_1, c_2, \ldots, c_t$ such that $\min\{a_1, \ldots, a_n\} > b_1 > b_2 > \cdots > b_{t+1}$ and $\max\{a_1, \ldots, a_n\} < c_1 < c_2 < \cdots < c_t$, construct the sequence

$$\hat{a} = (b_1, b_2, \ldots, b_{t+1}, c_1, c_2, \ldots, c_t, a_1, a_2, \ldots, a_n),$$

which we proceed to show is deranged. The sequence $\hat{a}$ has length $(t+1) + t + t^2 = (t+1)^2 = N$. Consider a monotone subsequence of $\hat{a}$ starting at some $b_i = \hat{a}_i$. If the subsequence contains a second term b_j, then it is necessarily decreasing, and thus a subsequence of $(b_1, b_2, \ldots, b_{t+1})$ (since each b_j is less than every a_k and every c_i by construction) and hence has length at most $t+1$. If the subsequence does not contain a second term b_j, but contains a term c_j, then it is necessarily increasing, so it is a subsequence of $(b_i, c_1, c_2, \ldots, c_t)$ (since each c_j is greater than every a_k by construction), and thus has length at most $t+1$. If the subsequence does not contain a second term b_j, nor any term c_j, then it consists of b_i followed by a decreasing subsequence of a; such a subsequence starting with b_i has length at most $1 + \max\{l_1(a), \ldots, l_n(a)\} \leq 1 + t$. Hence, $l_k(\hat{a}) \leq t + 1 = \left\lceil \sqrt{N+1-k} \right\rceil$ for $1 \leq k \leq t+1$. Similar consideration of a monotonic subsequence starting with some $c_i = \hat{a}_{t+1+i}$ shows that $l_k(\hat{a}) \leq \left\lceil \sqrt{N+1-k} \right\rceil$ for $t+2 \leq k \leq 2t+1$. For $2t+1 < k \leq n$, we have $l_k(\hat{a}) = l_{k-(2t+1)}(a) \leq \left\lceil \sqrt{n+1-(k-(2t+1))} \right\rceil = \left\lceil \sqrt{N+1-k} \right\rceil$ since a is deranged by hypothesis, hence $\hat{a}$ is a deranged sequence of length $N = (t+1)^2$.

To obtain a deranged sequence a of arbitrary (non-square) length n, it suffices to take the last n terms of a deranged sequence of length $t^2 \geq n$. Finally, to obtain a deranged permutation of $\{1, 2, \ldots, n\}$, let a be a length-n deranged sequence and let $\sigma \in S_n$ be the "sorting" permutation of a, so $a_{\sigma(1)} < a_{\sigma(2)} < \cdots < a_{\sigma(n)}$. The sequence $\sigma^{-1} = (\sigma^{-1}(1), \sigma^{-1}(2), \ldots, \sigma^{-1}(n))$ has the same relative ordering as the sequence $(a_1, a_2, \ldots, a_n)$, and thus σ^{-1} is a deranged permutation of $\{1, 2, \ldots, n\}$. To conclude the solution, let a be a deranged permutation of $\{1, 2, \ldots, 2018\}$. Then, $L(a) = M(2018) = 61\,440$ is minimum among all permutations.

Also solved by José Nieto (Venezuela), and the proposer. There were 2 incomplete or incorrect solutions.

A second-moment inequality when first moment is zero **October 2018**

2054. *Proposed by Florin Stanescu, Şerban Cioiculescu school, Găeşti, Romania.*

Let $f : [0,1] \to \mathbb{R}$ be differentiable with bounded derivative. If $\int_0^1 x f(x) dx = 0$, prove that

$$36 \cdot \left| \int_0^1 x^2 f(x) dx \right| \le \sup_{x \in [0,1]} |f'(x)|.$$

Solution by Lixing Han, University of Michigan-Flint, Flint, MI.
Integrating by parts, we have

$$\int_0^1 x^2 f'(x)\, dx = x^2 f(x)\big|_0^1 - 2 \int_0^1 x f(x)\, dx = f(1),$$

since $\int_0^1 x f(x)\, dx = 0$ by hypothesis. Integrating by parts again:

$$\int_0^1 x^2 f(x)\, dx = \frac{1}{3} x^3 f(x)\big|_0^1 - \frac{1}{3} \int_0^1 x^3 f'(x)\, dx = \frac{1}{3} f(1) - \frac{1}{3} \int_0^1 x^3 f'(x)\, dx.$$

Solving for $f(1)$ in this equation and combining with the first above, we obtain

$$\int_0^1 x^2 f(x)\, dx = \frac{1}{3} \int_0^1 x^2 f'(x)\, dx - \frac{1}{3} \int_0^1 x^3 f'(x)\, dx = \frac{1}{3} \int_0^1 (x^2 - x^3) f'(x)\, dx.$$

Therefore,

$$\begin{aligned} \left| \int_0^1 x^2 f(x)\, dx \right| &= \frac{1}{3} \left| \int_0^1 (x^2 - x^3) f'(x)\, dx \right| \le \frac{1}{3} \int_0^1 (x^2 - x^3) \left| f'(x) \right| dx \\ &\le \frac{1}{3} \int_0^1 (x^2 - x^3)\, dx \cdot \sup_{0 \le x \le 1} |f'(x)| = \frac{1}{36} \cdot \sup_{0 \le x \le 1} |f'(x)|. \end{aligned}$$

The inequality asserted in the statement of the problem follows immediately.

Also solved by Ulrich Abel (Germany), Michel Bataille (France), Robin Chapman (UK), Gary Chung, Michael P. Cohen, Robert Calcaterra, William Cowieson, Souvik Dey, Robert Doucette, Eugene Herman, Elgin Johnston, Koopa Koo (Hong Kong), Elias Lampakis (Greece), Kee-Wai Lau (Hong Kong), Joel Schlosberg, Ioannis Sfikas (Greece), Nicholas Singer, Albert Stadler (Switzerland), Michael Vowe (Switzerland), Scott Wolf, Shazeena Ashraf, Robert Summers, Braeden Duke & Matthew Cullum and the proposer.

Cyclic groups via characteristic subgroups **October 2018**

2055. *Proposed by Ioan Băetu, Botoşani, Romania.*

Let n be a cube-free positive integer. Assume that G is a finite group of order n such that for every subgroup H of G and every automorphism f of H, the equality $K = \{f(x) : x \in K\}$ holds for every subgroup K of H. Prove that G is cyclic.

Solution by Anthony J. Bevelacqua, University of North Dakota, Grand Forks, ND.
Suppose $x, y \in G$ satisfy $\langle x \rangle \cap \langle y \rangle = \{1\}$. By hypothesis, the conjugation automorphism $z \mapsto x^{-1} z x$ of G fixes $\langle y \rangle$, hence $x^{-1} y x \in \langle y \rangle$, and similarly $y^{-1} x^{-1} y \in \langle x^{-1} \rangle = \langle x \rangle$. It follows that $x^{-1} y^{-1} x y \in \langle x \rangle \cap \langle y \rangle = \{1\}$, so x and y commute.

Next, we show that, for any prime p dividing n, a Sylow p-subgroup P of G is cyclic. Denote by C_k the cyclic group of order $k \geq 1$. Since n is cube-free, P has order p or p^2; thus, P is isomorphic to one of the cyclic groups C_p, C_{p^2}, or the non-cyclic group $C_p \times C_p$. The subgroup $C_p \times \{1\}$ of $C_p \times C_p$ is not fixed by the automorphism $(x, y) \mapsto (y, x)$; thus, the hypothesis on G implies that P is not isomorphic to $C_p \times C_p$, so P is cyclic.

To conclude the proof, let $p_1, \ldots, p_r$ be the distinct primes dividing n. For $j = 1, \ldots, r$, let x_j be a generator of a Sylow p_j-subgroup of G. By the first Sylow theorem, we have $|x_1| \cdots |x_r| = n$. The elements $x_1, \ldots, x_r$ have pairwise coprime orders, hence generate groups with pairwise trivial intersection. By the argument in the first paragraph above, these elements commute pairwise, and furthermore $|x_1 \cdots x_r| = |x_1| \cdots |x_r| = n$. Hence, G is cyclic generated by $x_1 \cdots x_r$.

Editor's Note. Michael Reid pointed out that the hypothesis that G is finite may be relaxed to finitely generated (but not to infinitely generated). The conclusion that G is cyclic then follows from a more delicate argument using Baer's theorem.

Also solved by Robert Calcaterra, Robert Doucette, Abhay Goel, Koopa Koo (Hong Kong), José Nieto (Venezuela), Michael Reid, Nikhil Sahoo, Jacob Siehler, and the proposer.

Answers (Solutions to the Quickies from page 311.)

A1093. We have

$$2019^2 = 1480^2 + 969^2 + 555^2 + 485^2 + 455^2 + 300^2 + 200^2 + 185^2 + 150^2 + 100^2.$$

Therefore, for all $n > 0$, letting $m = n - 1 \geq 0$,

$$\begin{aligned}
2019^{2n} &= 2019^{2(m+1)} = 2019^2 \cdot 2019^{2m} \\
&= (1480^2 + 969^2 + 555^2 + 485^2 + 455^2 + 300^2 + 200^2 + 185^2 + 150^2 + 100^2) \cdot 2019^{2m} \\
&= (1480 \cdot 2019^m)^2 + (969 \cdot 2019^m)^2 + (555 \cdot 2019^m)^2 + (485 \cdot 2019^m)^2 \\
&\quad + (455 \cdot 2019^m)^2 + (300 \cdot 2019^m)^2 + (200 \cdot 2019^m)^2 + (185 \cdot 2019^m)^2 \\
&\quad + (150 \cdot 2019^m)^2 + (100 \cdot 2019^m)^2.
\end{aligned}$$

A1094. Implicit differentiation with respect to x gives $2(1 + y')\cos(x + y) + (1 - y')\sin(x - y) = 0$; hence,

$$\nu := \frac{y' + 1}{y' - 1} = \frac{\sin(x - y)}{2\cos(x + y)}.$$

Using trigonometric identities and the relation $2\sin(x + y) - \cos(x - y) = 1$, we obtain

$$\begin{aligned}
\nu^2 &= \frac{\sin^2(x - y)}{4\cos^2(x + y)} = \frac{[1 + \cos(x - y)][1 - \cos(x - y)]}{4[1 + \sin(x + y)][1 - \sin(x + y)]} \\
&= \frac{2\sin(x + y)}{2[1 + \sin(x + y)]} \cdot \frac{1 - \cos(x - y)}{2 - 2\sin(x + y)} = \frac{\sin(x + y)}{1 + \sin(x + y)}.
\end{aligned}$$

Thus, at the double point $(\pi/4, \pi/4)$, we have $\nu^2 = \sin(\pi/2)/[1 + \sin(\pi/2)] = 1/2$, so $\nu = \pm 1/\sqrt{2}$. Either of the tangent line slopes $m = y'$ at the double point is related to the respective inclination angle θ by $m = \tan\theta$, while $\nu = (\tan\theta + 1)/(\tan\theta - 1) = \tan(-\theta - \pi/4)$. It follows that the angle sought, equal to the difference of the inclination angles θ_1, θ_2, is equal to $\theta_2 - \theta_1 = (-\pi/4 - \theta_1) - (-\pi/4 - \theta_2) = \arctan(1/\sqrt{2}) - \arctan(-1/\sqrt{2}) = 2\arctan(1/\sqrt{2}) \approx 70.53°$.