The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://maa.tandfonline.com/loi/uamm20

Problems and Solutions

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PROBLEMS AND SOLUTIONS

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with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by November 30, 2019, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12125. *Proposed by James Propp, University of Massachusetts, Lowell, MA.*

(a) In the picture at right, nine equally spaced points on a circle are joined by nine chords, forming seven triangles. Show that the sum of the areas of the three outermost black triangles plus the area of the innermost (equilateral) black triangle equals the sum of the areas of the other three triangles.

(b) Part **(a)** can be phrased as the assertion that a certain self-intersecting 9-gon has signed area zero. For what values of n does there exist a self-intersecting n-gon of signed area zero whose vertices coincide with the vertices of a regular n-gon?

12126. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bîrlad, Romania.* Let $P(n)$ be the greatest prime divisor of the positive integer n. Prove that $P(n^2 - n + 1) < P(n^2 + n + 1)$ and $P(n^2 - n + 1) > P(n^2 + n + 1)$ each hold for infinitely many positive integers n.

12127. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\int_0^1 \left(\frac{\mathrm{Li}_2(1) - \mathrm{Li}_2(x)}{1 - x} \right)^2 dx,$$

where Li_2 denotes the dilogarithm function, defined by $\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} z^k/k^2$.

12128. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let F_n be the nth Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Find, in terms of n, the number of trailing zeros in the decimal representation of F_n.

doi.org/10.1080/00029890.2019.1621132

12129. *Proposed by Hideyuki Ohtsuka, Saitama, Japan.* Compute

$$\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2-\sqrt{2+\cdots}}}}},$$

where the sequence of signs consists of $n-1$ plus signs followed by a minus sign and repeats with period n.

12130. *Proposed by Dan Ştefan Marinescu, Hunedoara, Romania, and Mihai Monea, Deva, Romania.* Let P be a point in the interior of triangle ABC. Suppose that the lines AP, BP, and CP intersect the circumcircle of ABC again at A', B', and C', respectively. Prove

$$\frac{\mathrm{S}(BPC)}{AP}+\frac{\mathrm{S}(APC)}{BP}+\frac{\mathrm{S}(APB)}{CP}\geq\frac{\mathrm{S}(BPC)}{A'P}+\frac{\mathrm{S}(APC)}{B'P}+\frac{\mathrm{S}(APB)}{C'P},$$

where $S(XYZ)$ denotes the area of triangle XYZ.

12131. *Proposed by Michael Maltenfort, Northwestern University, Evanston, IL.* Let m and n be positive integers with $n \geq 2$. Suppose that U is an open subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ is continuously differentiable. Let K be the set of all $x \in U$ such that the derivative $Df(x)$, as a linear transformation, has rank less than n. Prove that if $f(K)$ is countable, $U \setminus K \neq \emptyset$, and $f(U)$ is closed, then $f(U) = \mathbb{R}^n$.

SOLUTIONS

Cycle of Powers

11665 [2012, 669]. *Proposed by Raitis Ozols, student, University of Latvia, Riga, Latvia.* Let $a = (a_1, \ldots, a_n)$, where $n \geq 2$ and each a_j is a positive real number. Let $S(a) = a_1^{a_2} + \cdots + a_{n-1}^{a_n} + a_n^{a_1}$.
(a) Prove that $S(a) > 1$.
(b) Prove that for all $\epsilon > 0$ and $n \geq 2$ there exists a of length n with $S(a) < 1 + \epsilon$.

Solution by Traian Viteam, Punta Arenas, Chile. First, we prove the result for $n = 2$. We show that if $a, b > 0$, then $a^b + b^a > 1$. If one of a and b is at least 1, this is clear, so we henceforth assume $0 < a, b < 1$. From Bernoulli's inequality, we have

$$a^{1-b} = (1 + (a-1))^{1-b} < 1 + (1-b)(a-1) = a + b - ab.$$

Hence $a^b > \frac{a}{a+b-ab}$. Similarly, $b^a > \frac{b}{a+b-ab}$, so

$$a^b + b^a > \frac{a}{a+b-ab} + \frac{b}{a+b-ab} = \frac{a+b}{a+b-ab} > 1.$$

For $n \geq 3$, we may assume by cyclic symmetry that $a_1 = \max\{a_1, \ldots, a_n\}$. Again, when $a_1 \geq 1$ we are obviously done, so we may assume that a_i is in $(0, 1)$ for all i. We then have

$$S(a) > a_1^{a_2} + a_2^{a_3} \geq a_1^{a_2} + a_2^{a_1} > 1,$$

where the final step is the case $n = 2$.

For part (b), let ϵ be an arbitrary positive constant. Choose $a_n = 1$. We define $a_{n-1}, \ldots, a_1$ inductively. Assume that we have defined positive reals $a_{n-k}, \ldots, a_n$. Since

$\lim_{x\to 0} x^{a_{n-k}} = 0$, we can choose a_{n-k-1} small enough so $a_{n-k-1}^{a_{n-k}} < \epsilon/(n-1)$. Once we have defined $a_1, \ldots, a_n$ in this way,

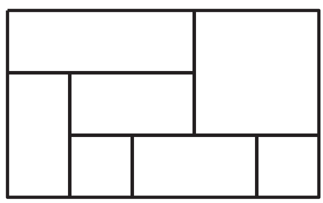
$$S(a) < (n-1)\frac{\epsilon}{n-1} + 1 = 1 + \epsilon.$$

Editorial comment. The editors regret the delay in the appearance of this solution. The case $n = 2$ of this inequality, from which the general case easily follows as shown above, has appeared before. For example, it is inequality 3.6.38 on page 281 in D. S. Mitrinović, (1970), *Analytic Inequalities*, Berlin: Springer-Verlag.

Also solved by K. F. Andersen (Canada), G. Apostolopoulos (Greece), R. Boukharfane (France), N. Caro (Brazil) and O. López (Colombia), H. Chen, J. Chun (South Korea), P. P. Dálya (Hungary), V. De Angelis, A. Ercan (Turkey), D. Fleischman, A. Habil (Syria), E. A. Herman, Y. J. Ionin, H. Katsuura & E. Schmeichel, O. Kouba (Syria), J. Li, M. Omarjee (France), P. Perfetti (Italy), M. A. Prasad (India), R. Stong, M. Vowe (Switzerland), GCHQ Problem Solving Group (U. K.), and the proposer.

Tight Pavings by Integer Rectangles

12005 [2018, 755]. *Proposed by Donald E. Knuth, Stanford, CA.* A *tight m-by-n paving* is a decomposition of an m-by-n rectangle into $m + n - 1$ rectangular tiles with integer sides such that each of the $m - 1$ horizontal lines and $n - 1$ vertical lines within the rectangle is part of the boundary of at least one tile. For example, one of the 1071 tight 3-by-5 pavings is pictured here:



Let $a_{m,n}$ denote the number of tight m-by-n pavings.
(a) Determine $a_{3,n}$ as a function of n.
(b) Show for $m \geq 3$ that $\lim_{n\to\infty} a_{m,n}/m^n$ exists, and compute its value.

Composite solution by Richard Stong, Center for Communications Research, San Diego, CA, Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy, and O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. The answers are **(a)** $a_{3,n} = \frac{27}{4}3^n - 20 \cdot 2^n + n^2 + \frac{13}{2}n + \frac{53}{4}$ and **(b)** $\lim_{n\to\infty} a_{m,n}/m^n = m^{2m-1}/(m!)^2$.

A *paving* is any decomposition as described in the problem statement, except for dropping the requirement that the number of tiles is $m + n - 1$. We show that the minimum number of tiles in a paving is $m + n - 1$. The pavings achieving this minimum number of tiles are called *tight*. For convenience, we use *gridline* to mean one of the $m + n - 2$ horizontal or vertical lines that cross the rectangle internally at a positive integer distance from the sides. An *edge* is a side of any rectangle in the paving. A *segment* is a maximal connected union of edges along a single gridline. The condition for a paving is that every gridline contains at least one edge.

Lemma. *In a tight paving, no vertical segment crosses a horizontal segment (at an internal point of both), and the edges on any gridline form a single segment.*

Proof. In any paving, say that a tile T *witnesses* a horizontal gridline h if it is the leftmost tile whose top is on h and *witnesses* a vertical gridline v if it is the highest tile whose left side is on v. Note that (1) the tile U at the upper left corner witnesses no gridline, (2) each gridline is witnessed by exactly one tile, and (3) no tile witnesses more than one gridline (the segments at the top and left of a tile T witnessing horizontal and vertical gridlines would not continue leftward or upward, preventing the tiling from being completed).

These three observations imply that every paving has at least $m + n - 1$ tiles, so tight pavings are those with the fewest tiles, and every tile other than U in such a paving witnesses exactly one gridline. If two segments cross, then the crossing point is a corner of four tiles, and the one on the lower right of these four would witness no gridline.

For the second statement, suppose by symmetry that a horizontal gridline h contains more than one segment. Let T_1 be the tile witnessing h, and let edge E be a leftmost edge on the next segment along h. Since the segment containing E does not extend leftward, the portion of h to the left of E is internal to some tile T_2. Now the left endpoint of E is the upper left corner of a tile T_3 that does not witness the gridline for its top or left edge, contradicting that every tile other than U witnesses a gridline. ■

(a) An m-by-n rectangle has $m - 1$ horizontal gridlines. By the lemma, every tight paving contains exactly one segment on each horizontal gridline. Let H_j denote the interval obtained by projecting the segment from the gridline at height j onto the horizontal axis.

For $m = 3$, consider first the case where $H_1 = H_2$ (as in Figure 1, where $x_2 = 3$). Since neither horizontal segment extends and each gridline contains a single segment, there are no horizontal edges not on these segments, so all the tiles to the left and right of these horizontal segments have width 1 and height 3.

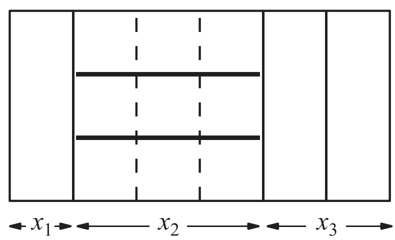


Figure 1. Horizontal segments of equal extent.

Now consider the vertical segments between the endpoints of the two horizontal segments. Since segments cannot cross, each of these $x_2 - 1$ vertical gridlines contains a segment of length one in one of three possible places, and all such choices yield pavings. Each insertion of a vertical segment increases the number of tiles by 1, so there are $3 + x_2 - 1$ tiles along the horizontal segments and $n - x_2$ tiles outside them, totaling $n + 2$.

Letting N be the number of tight pavings in this case, we have $N = \sum_{x \in P_1} 3^{x_2 - 1}$, where P_1 is the set of nonnegative integer triples (x_1, x_2, x_3) with sum n such that $x_2 \geq 1$. Using $[z^n]f(z)$ to mean the coefficient of z^n in $f(z)$, we have

$$N = [z^n] \sum_{x_1 \geq 0} z^{x_1} \sum_{x_2 \geq 1} \frac{1}{3}(3z)^{x_2} \sum_{x_3 \geq 0} z^{x_3} = [z^n] \frac{1}{1-z} \frac{z}{1-3z} \frac{1}{1-z}.$$

There are four other cases, illustrated in Figure 2. The intervals H_1 and H_2 may have no positive overlap, have overlap without containment, exhibit strict containment at both ends, or be equal at one end. Due to reflections, the first three of these cases may occur in two ways, the last in four ways.

These cases lead, in the same way as above, to four generating functions. For each case, the contribution to $a_{3,n}$ will be a sum over nonnegative choices of the variables summing to n, where variables giving lengths of portions of the horizontal segments must be positive. For a variable x measuring a portion covered by both horizontal segments, the factor in the

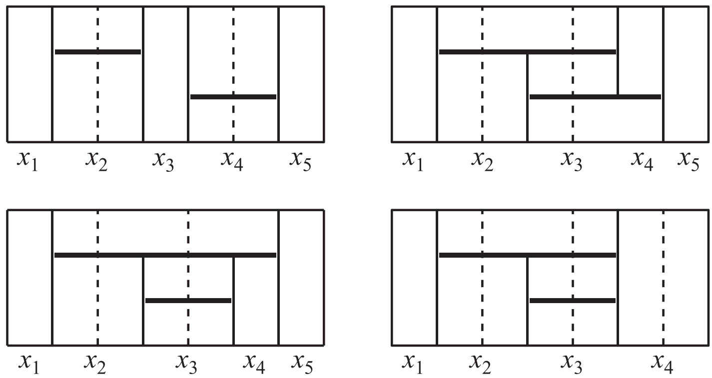


Figure 2. The remaining four cases.

number of choices is 3^{x-1}; for a portion covered by only one of the horizontal segments, it is 2^{x-1} (again because no two segments cross). We obtain the following contributions.

Case	#Tilings	Generating Function
0 (Figure 1)	$\sum 3^{x_2-1}$	$\frac{z}{(1-z)^2(1-3z)}$
1 (Figure 2)	$2\sum 2^{x_2-1}2^{x_4-1}$	$\frac{2z^2}{(1-z)^3(1-2z)^2}$
2 (Figure 2)	$2\sum 2^{x_2-1}3^{x_3-1}2^{x_4-1}$	$\frac{2z^3}{(1-z)^2(1-2z)^2(1-3z)}$
3 (Figure 2)	$2\sum 2^{x_2-1}3^{x_3-1}2^{x_4-1}$	$\frac{2z^3}{(1-z)^2(1-2z)^2(1-3z)}$
4 (Figure 2)	$4\sum 2^{x_2-1}3^{x_3-1}$	$\frac{4z^2}{(1-z)^2(1-2z)(1-3z)}$

The sum of the five rational functions is $\frac{z(1+3z)}{(1-z)^3(1-2z)(1-3z)}$, which has partial fraction expansion

$$\frac{27/4}{1-3z} - \frac{20}{1-2z} + \frac{2}{(1-z)^3} + \frac{7/2}{(1-z)^2} + \frac{31/4}{1-z}.$$

Thus

$$\begin{aligned} a_{3,n} &= \frac{27}{4}3^n - 20\cdot 2^n + 2\binom{n+2}{2} + \frac{7}{2}(n+1) + \frac{31}{4} \\ &= \frac{27}{4}3^n - 20\cdot 2^n + n^2 + \frac{13}{2}n + \frac{53}{4}. \end{aligned}$$

(b) Let $\lambda_m = \lim_{n\to\infty} a_{m,n}/m^n$. Asymptotically, we can restrict to tight pavings where $H_1, \dots, H_{m-1}$ have a common subinterval of positive length. The reason is that the number of tight pavings yielding no such overlap is less than $n^{2(m-1)}(m-1)^{n-1}$ (and the ratio of this to m^n tends to 0 as $n \to \infty$). To see this, note first that each of $H_1, \dots, H_{m-1}$ can be specified in fewer than n^2 ways. For the vertical segments, since each gridline has one segment and they don't cross, the lack of a common horizontal overlap implies that there are at most $m-1$ ways to place each vertical segment (extending part **(a)**). Let $\hat{a}_{m,n}$ be the number of tight pavings of the m-by-n rectangle where $H_1, \dots, H_{m-1}$ have a common overlap.

For any paving counted by $\hat{a}_{m,n}$, we partition the interval $[0, n]$ into three subintervals of lengths k, d, and l, where d is the positive length of $\bigcap H_i$, k is the length of the part of

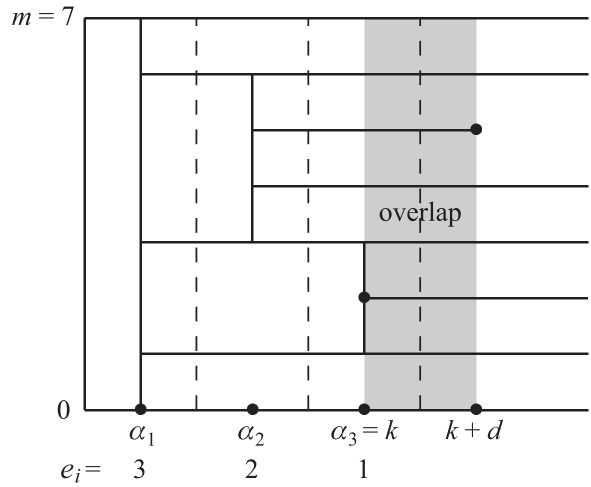


Figure 3. Part of a tight paving with $(m, k) = (7, 5)$ and multiset $[1^3, 3^2, 5^1]$.

the gridlines to its left, and l is the remaining length to the right. Some H_i starts at k, and some H_i ends at $k + d$.

The left ends of $H_1, \dots, H_{m-1}$ form a multiset of size $m - 1$ from $\{0, \dots, k\}$, using k at least once. With $H_i = [a_i, b_i]$, let $\alpha_1, \dots, \alpha_r$ in increasing order be the values occurring as some a_i, having multiplicities $e_1, \dots, e_r$. Write the multiset as $[\alpha_1^{e_1}, \dots, \alpha_r^{e_r}]$.

The key restriction on the list $a_1, \dots, a_{m-1}$ is that if $a_i = a_j = \beta$ with $i < j$, then $a_t \ge \beta$ for all t with $i < t < j$. Since H_i and H_j do not extend leftward of a_i, the points (β, i) and (β, j) lie on vertical edges. Since each vertical gridline contains only one segment, (β, t) is internal to the vertical segment at horizontal position β. Since $b_t \ge k + d > \beta$ and segments cannot cross, $a_t \ge \beta$.

With this restriction, we count the ways to form the list $a_1, \dots, a_{m-1}$ using the multiset $[\alpha_1^{e_1}, \dots, \alpha_r^{e_r}]$. The restriction implies that the copies of α_j in $a_1, \dots, a_{m-1}$ occupy e_j consecutive blank positions among the $m - 1 - \sum_{i=j+1}^{r} e_i$ blank positions left by placing the copies of all α_i with $i > j$. Since e_j copies of α_j must be placed, there are $m - \sum_{i=j}^{r} e_i$ possible places to start the copies of α_j, regardless of how the larger values were placed. Since $\sum_{i=1}^{r} e_i = m - 1$, the number of configurations of the left endpoints corresponding to the given multiset is $\prod_{j=1}^{r-1}\left(1 + \sum_{i=1}^{j} e_i\right)$.

Between the horizontal positions α_j and α_{j+1} are $\alpha_{j+1} - \alpha_j - 1$ vertical gridlines. No horizontal segments end at these gridlines. Hence the segment on each such vertical gridline is a single edge joining two of the horizontal segments (including the top and bottom edges) that start at position α_j or earlier. That gives $1 + \sum_{i=1}^{j} e_i$ choices for the vertical segment.

After forming the list $a_1, \dots, a_{m-1}$ and placing the vertical segments, we have $\prod_{j=1}^{r-1}\left(1 + \sum_{i=1}^{j} e_i\right)^{\alpha_{j+1}-\alpha_j}$ ways to form the left part of the paving from the given multiset. Let $s_{m,k}$ denote the sum of these quantities over all multisets of size $m - 1$ chosen from $\{0, \dots, k\}$.

We can write a multiset $[\alpha_1^{e_1}, \dots, \alpha_r^{e_r}]$ as $[0^{f_0}, \dots, k^{f_k}]$ by including the multiplicities of the unused elements, which equal 0. We then have

$$\prod_{j=1}^{r-1}\left(1 + \sum_{i=1}^{j} e_i\right)^{\alpha_{j+1}-\alpha_j} = \prod_{l=0}^{k-1}\left(1 + \sum_{i=0}^{l} f_i\right) = \prod_{l=0}^{k-1} c_l,$$

where $c_l = 1 + \sum_{i=0}^{l} f_i$. The list $c_0, \ldots, c_{k-1}$ is a weakly increasing integer list with values between 1 and $m-1$. Over all choices of the multiset $[\alpha_1^{e_1}, \ldots, \alpha_r^{e_r}]$ from $\{0, \ldots, k\}$, we obtain all such lists. That is, $s_{m,k} = \sum_{c \in L_{m,k}} \prod_{l=0}^{k-1} c_l$, where $L_{m,k}$ is the set of all k-element nonnegative integer lists c such that $1 \le c_0 \le \cdots \le c_{k-1} \le m-1$.

Within the central overlap portion, each vertical gridline must have a single edge of length 1; there are m^{d-1} ways to place these. The right portion of the paving is constructed symmetrically to the left portion, over an interval of length $n-d-k$. Thus

$$\hat{a}_{m,n} = \sum_{d=1}^{n} \sum_{k=0}^{n-d} s_{m,k} s_{m,n-d-k} m^{d-1}.$$

Replacing d with $n-k-l$, we write

$$\lambda_m = \lim_{n\to\infty} \frac{\hat{a}_{m,n}}{m^n} = \lim_{n\to\infty} \frac{1}{m} \sum_{k+l<n} \frac{s_{m,k}}{m^k} \frac{s_{m,l}}{m^l}.$$

The key now is to replace the sum over a triangle of values with a sum over a square of values, separating the sums over k and l. We have

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \sum_{l=0}^{\lfloor (n-1)/2 \rfloor} \frac{s_{m,k}}{m^k} \frac{s_{m,l}}{m^l} \le \sum_{k+l<n} \frac{s_{m,k}}{m^k} \frac{s_{m,l}}{m^l} \le \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \frac{s_{m,k}}{m^k} \frac{s_{m,l}}{m^l}.$$

As $n \to \infty$, the upper and lower bounds are the same; hence the limit of the middle expression must be the same as the limit of the outer expressions.

Thus $\lambda_m = \frac{1}{m} \left(\sum_{k=0}^{\infty} s_{m,k}/m^k \right)^2$. To turn this into the desired limit $m^{2m-1}/(m!)^2$, it suffices to prove $\sum_{k=0}^{\infty} s_{m,k}/m^k = m^{m-1}/(m-1)!$. To do this, we compute

$$\sum_{k=0}^{\infty} \frac{s_{m,k}}{m^k} = \sum_{k=0}^{\infty} \sum_{c \in L_{m,k}} \prod_{i=0}^{k-1} \frac{c_i}{m} = \prod_{q=0}^{m-1} \sum_{t=0}^{\infty} \left(\frac{q}{m} \right)^t = \prod_{q=0}^{m-1} \frac{1}{1-q/m} = \frac{m^{m-1}}{(m-1)!}.$$

To justify the second equality here, note that the double sum $\sum_{k=0}^{\infty} \sum_{c \in L_{m,k}}$ encounters every multiset of values chosen from $\{0, \ldots, m-1\}$. Over the full sum, any multiplicity of a given value q is grouped with all possible multiplicities of other values. Hence we can regroup the terms by the values, leading to the product of infinite sums for each of the values.

Editorial comment. The sequence in part **(a)** appears as sequence A285361 at oeis.org.

Also solved by H. K. Pillai (India) and M. A. Prasad (India; part (a) only).

A Hyperbolic Limit of Trigonometric Matrices

12014 [2018, 81]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let a, b, c, and d be real numbers with $bc > 0$. Calculate

$$\lim_{n\to\infty} \begin{bmatrix} \cos(a/n) & \sin(b/n) \\ \sin(c/n) & \cos(d/n) \end{bmatrix}^n.$$

Solution by Tamas Wiandt, Rochester Institute of Technology, Rochester, NY. The limit is

$$\begin{bmatrix} \cosh\sqrt{bc} & \sqrt{b/c}\,\sinh\sqrt{bc} \\ \sqrt{c/b}\,\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{bmatrix}.$$

Letting

$$A_n = \begin{bmatrix} \cos(a/n) - 1 & \sin(b/n) \\ \sin(c/n) & \cos(d/n) - 1 \end{bmatrix},$$

we have

$$\begin{bmatrix} \cos(a/n) & \sin(b/n) \\ \sin(c/n) & \cos(d/n) \end{bmatrix} = I + A_n,$$

where I is the 2×2 identity matrix. When n is large enough, $\|A_n\| < 1$ and

$$\log(I + A_n) = A_n - \frac{1}{2}A_n^2 + \frac{1}{3}A_n^3 - \frac{1}{4}A_n^4 + \cdots .$$

Since

$$A_n = \begin{bmatrix} O(1/n^2) & b/n + O(1/n^3) \\ c/n + O(1/n^3) & O(1/n^2) \end{bmatrix},$$

we have $\log(I + A_n) = A_n + O(1/n^2)$ and $n \log(I + A_n) = nA_n + O(1/n)$. Since

$$\lim_{n\to\infty} n \log(I + A_n) = \lim_{n\to\infty} nA_n = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix},$$

we obtain

$$\lim_{n\to\infty} \begin{bmatrix} \cos(a/n) & \sin(b/n) \\ \sin(c/n) & \cos(d/n) \end{bmatrix}^n = \lim_{n\to\infty} \exp(n \log(I + A_n)) = \exp\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right).$$

If $bc > 0$, then the matrix $\left[\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right]$ has distinct eigenvalues $\sqrt{bc}$ and $-\sqrt{bc}$, and

$$\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{b} & \sqrt{b} \\ \sqrt{c} & -\sqrt{c} \end{bmatrix} \begin{bmatrix} \sqrt{bc} & 0 \\ 0 & -\sqrt{bc} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{b}} & \frac{1}{2\sqrt{c}} \\ \frac{1}{2\sqrt{b}} & -\frac{1}{2\sqrt{c}} \end{bmatrix},$$

where $b, c > 0$. Thus

$$\exp\left(\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{b} & \sqrt{b} \\ \sqrt{c} & -\sqrt{c} \end{bmatrix} \begin{bmatrix} e^{\sqrt{bc}} & 0 \\ 0 & e^{-\sqrt{bc}} \end{bmatrix} \begin{bmatrix} \frac{1}{2\sqrt{b}} & \frac{1}{2\sqrt{c}} \\ \frac{1}{2\sqrt{b}} & -\frac{1}{2\sqrt{c}} \end{bmatrix}$$

$$= \begin{bmatrix} \cosh\sqrt{bc} & \sqrt{b/c}\,\sinh\sqrt{bc} \\ \sqrt{c/b}\,\sinh\sqrt{bc} & \cosh\sqrt{bc} \end{bmatrix}.$$

The case where $b, c < 0$ is similar.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), R. Chapman (U. K.), H. Chen, G. Fera (Italy), D. Fleischman, C. Georghiou (Greece), J. Grivaux (France), A. Goel, E. A. Herman, Y. Hu (China), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), A. Minasyan (Russia), R. Nandan, M. Omarjee, F. Perdomo & Á. Plaza (Spain), K. Schilling, J. Singh (India), J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), N. Thornber, E. I. Verriest, Z. Vörös (Hungary), A. Wentworth, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.

A Symmetric Sum

12016 [2018, 81]. *Proposed by Hideyuki Ohtsuka, Saitama, Japan, and Roberto Tauraso, Università di Roma "Tor Vergata," Rome, Italy.* For nonnegative integers m, n, r, and s, prove

$$\sum_{k=0}^{s} \binom{m+r}{n-k}\binom{r+k}{k}\binom{s}{k} = \sum_{k=0}^{r} \binom{m+s}{n-k}\binom{s+k}{k}\binom{r}{k}.$$

Solution by Nicole Grivaux, Paris, France. Let $A(r, s)$ be the left side of the equation to be proved. Throughout, we use the convention that $\binom{a}{b} = 0$ whenever $b > a \geq 0$. By the Vandermonde identity and symmetry,

$$\binom{r+k}{k} = \sum_{i=0}^{r} \binom{r}{i}\binom{k}{k-i} = \sum_{i=0}^{r} \binom{r}{i}\binom{k}{i}.$$

Hence

$$\begin{aligned} A(r, s) &= \sum_{i=0}^{r} \binom{r}{i} \sum_{k=0}^{s} \binom{m+r}{n-k}\binom{s}{k}\binom{k}{i} \\ &= \sum_{i=0}^{r} \binom{r}{i} \sum_{k=0}^{s} \binom{m+r}{n-k}\binom{s}{i}\binom{s-i}{k-i} \\ &= \sum_{i=0}^{\min(r,s)} \binom{r}{i}\binom{s}{i} \sum_{k=0}^{s} \binom{m+r}{n-k}\binom{s-i}{k-i} \\ &= \sum_{i=0}^{\min(r,s)} \binom{r}{i}\binom{s}{i}\binom{m+r+s-i}{n-i}. \end{aligned}$$

The second equality follows from $\binom{s}{k}\binom{k}{i} = \binom{s}{i}\binom{s-i}{k-i}$, while the fourth is another application of the Vandermonde identity. Since the final form is symmetric in r and s, we conclude $A(r, s) = A(s, r)$, which is the desired equality.

Also solved by U. Abel (Germany), H. Almusawa & N. Alobaidan & R. Jacobs & D. Nuraliyev & J. Shive & M. Apagodu, T. Amdeberhan & V. H. Moll, R. Chapman (U. K.), S. B. Ekhad, R. Evans, G. Fera (Italy), D. Fleischman, O. Kouba (Syria), P. Lalonde (Canada), O. P. Lossers (Netherlands), J. C. Smith, A. Stadler (Switzerland), R. Stong, M. Wildon (U. K.), GCHQ Problem Solving Group (U. K.), and the proposers.

Euler's Totient is Sparse

12021 [2018, 179]. *Proposed by Omar Sonebi, Lycée Technique, Settat, Morocco.* Let ϕ be the Euler totient function. Given $a \in \mathbb{Z}^+$ and $b \in \mathbb{Z}^+$, show that there exists $n \in \mathbb{Z}^+$ such that $an + b$ is not in the range of ϕ.

Solution by Li Zhou, Polk State College, Winter Haven, FL. Let $d = \gcd(a, b)$, with $a = dr$ and $b = ds$. Set $t = r\prod_{i=1}^{d}(is+1)$; note that t is relatively prime to s. By Dirichlet's theorem, there is a prime p of the form $tm + s$ for some $m \in \mathbb{Z}^+$. Let $n = tm/r$. We claim that $an + b$, which equals dp, is not in the range of ϕ. If $dp = \phi(N)$ for some $N \in \mathbb{Z}^+$ having prime factorization $\prod_{j=1}^{k} p_j^{e_j}$, then $dp = \prod_{j=1}^{k} p_j^{e_j-1}(p_j - 1)$. Since $p - 1 > d$, we conclude that p is a factor of $p_i - 1$ for some i. Now $p_i = qp + 1$ for some q with $1 \leq q \leq d$. Since $qp + 1 = q(tm + s) + 1 = (qs + 1) + qmr\prod_{i=1}^{d}(is + 1)$, this requires $qp + 1$ to have $qs + 1$ as a proper factor, so $qp + 1$ cannot be prime. This contradiction completes the proof of the claim.

Editorial comment. Souvik Dey and Celia Schacht noted that the claim immediately follows from the more general result of S. S. Pillai (1929), On some functions connected with $\phi(n)$, *Bull. Amer. Math. Soc.* 35: 832–836, which implies that if $N(n)$ is the number of positive integers up to n that are in the range of ϕ, then $\lim_{n\to\infty} N(n)/n = 0$.

Also solved by S. Chandrasekhar, A. Cheraghi (Canada), S. Dey (India), G. Fera (Italy), D. Fleischman, K. Gatesman, Y. J. Ionin, J. Kim (South Korea), O. P. Lossers (Netherlands), M. Omarjee (France), M. Reid, C. Schacht, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), AN-anduud Problem Solving Group (Mongolia), GCHQ Problems Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.