The American Mathematical Monthly

ISSN: 0002-9890 (Print) 1930-0972 (Online) Journal homepage: https://maa.tandfonline.com/loi/uamm20

Problems and Solutions

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PROBLEMS AND SOLUTIONS

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with the collaboration of Paul Bracken, Ezra A. Brown, Zachary Franco, Christian Friesen, László Lipták, Rick Luttmann, Frank B. Miles, Lenhard Ng, Kenneth Stolarsky, Richard Stong, Daniel Velleman, Stan Wagon, Elizabeth Wilmer, Fuzhen Zhang, and Li Zhou.

Proposed problems should be submitted online at
americanmathematicalmonthly.submittable.com/submit.
Proposed solutions to the problems below should be submitted by April 30, 2020, via the same link. More detailed instructions are available online. Proposed problems must not be under consideration concurrently at any other journal nor be posted to the internet before the deadline date for solutions. An asterisk (*) after the number of a problem or a part of a problem indicates that no solution is currently available.

PROBLEMS

12146. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Let n be an integer greater than 1, and let $[n]$ denote $\{1, \dots, n\}$ as usual. Let $\pi_1, \pi_2, \dots, \pi_{n!}$ be a list of the $n!$ permutations of $[n]$, ordered lexicographically with respect to the word $\pi_k(1)\pi_k(2)\cdots\pi_k(n)$. For example, with $n = 3$, the 6 words in order are 123, 132, 213, 231, 312, and 321.
(a) For $1 \le k < n!$, let ψ_k be the permutation of $[n]$ defined by $\psi_k(i) = j$ if and only if $\pi_k(i) = \pi_{k+1}(j)$. What is the cardinality of $\{\psi_k : 1 \le k < n!\}$?
(b) For $1 \le k < n!$, let φ_k be the permutation of $[n]$ defined by $\varphi_k(\pi_k(j)) = \pi_{k+1}(j)$. What is the cardinality of $\{\varphi_k : 1 \le k < n!\}$?

12147. *Proposed by Luis González, Houston, TX, and Tran Quang Hung, Hanoi National University, Hanoi, Vietnam.* Let $ABCD$ be a quadrilateral that is not a parallelogram. The *Newton line* of $ABCD$ is the line that connects the midpoints of the diagonals AC and BD. Let X be the intersection of the perpendicular bisectors of AB and CD, and let Y be the intersection of the perpendicular bisectors of BC and DA. Prove that XY is perpendicular to the Newton line of $ABCD$.

12148. *Proposed by Tibor Beke, University of Massachusetts, Lowell, MA.* Let p be a prime number, and let f be a symmetric polynomial in $p-1$ variables with integer coefficients. Suppose that f is homogeneous of degree d and that $p-1$ does not divide d. Prove that p divides $f(1, 2, \dots, p-1)$.

12149. *Proposed by Mohammadhossein Mehrabi, Sala, Sweden.* Let Γ be the gamma function, defined by $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$. Prove

$$x^x y^y \, \Gamma\left(\frac{x+y}{2}\right)^2 \le \left(\frac{x+y}{2}\right)^2 \Gamma(x)\Gamma(y)$$

for all positive real numbers x and y.

12150. *Proposed by Péter Kórus, University of Szeged, Szeged, Hungary.* Let $X_0, \dots, X_n$ be independent random variables, each distributed uniformly on $[0, 1]$. Calculate the expected value of $\min_{1 \le k \le n} |X_0 - X_k|$.

doi.org/10.1080/00029890.2019.1664219

12151. *Proposed by Leonard Giugiuc and Cezar Alexandru Trancanau, Drobeta Turnu Severin, Romania, and Michael Rozenberg, Tel Aviv, Israel.* Let A, B, C, and M be points in the plane with A, B, and C distinct. Let A', B', and C' be the reflections through M of A, B, and C, respectively. Determine the minimum value of $AB'/AB + BC'/BC + CA'/CA$ under the constraint that

(a) A, B, C, and M are collinear.

(b) A, B, and C are not collinear.

12152. *Proposed by George Stoica, Saint John, NB, Canada.* Let f be a twice differentiable real-valued function on $[0, \infty)$ such that $f(0) = 1$, $f'(0) = 0$, and $f(x)f''(x) = 1$ for all positive x. Find $\lim_{x\to\infty} f(x)/(x\sqrt{\ln x})$.

SOLUTIONS

An Integral Involving Fractional Parts

12031 [2018, 277]. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* **(a)** Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\} dx\, dy = 1 - \gamma,$$

where $\{a\}$ denotes the fractional part of a, and γ is Euler's constant.

(b) Let k be a nonnegative integer. Prove

$$\int_0^1 \int_0^1 \left\{ \frac{x}{1-xy} \right\}^k dx\, dy = \int_0^1 \left\{ \frac{1}{x} \right\}^k dx.$$

Solution by Eugene A. Herman, Grinnell College, Grinnell, IA. We begin with (**b**). More generally, we prove

$$\int_0^1 \int_0^1 f\left(\left\{ \frac{x}{1-xy} \right\} \right) dx\, dy = \int_0^1 f\left(\left\{ \frac{1}{x} \right\} \right) dx$$

for any bounded measurable function f on $[0, 1]$. To prove this, we first change variables to $u = 1/x - y$ and $v = y$. Thus $x = (u+v)^{-1}$ and $y = v$, and so we have $dx\, dy = (u+v)^{-2}\, dv\, du$. Since $u + v = 1/x \ge 1$, the new domain of integration consists of the two regions $\{(u, v) : 1 \le u < \infty, 0 \le v \le 1\}$ and $\{(u, v) : 0 \le u \le 1, 1 - u \le v \le 1\}$. Therefore

$$\begin{aligned}
&\int_0^1 \int_0^1 f\left(\left\{ \frac{x}{1-xy} \right\} \right) dx\, dy \\
&\quad = \int_1^\infty \int_0^1 f\left(\left\{ \frac{1}{u} \right\} \right) \frac{1}{(u+v)^2}\, dv\, du + \int_0^1 \int_{1-u}^1 f\left(\left\{ \frac{1}{u} \right\} \right) \frac{1}{(u+v)^2}\, dv\, du \\
&\quad = \int_1^\infty f\left(\left\{ \frac{1}{u} \right\} \right) \frac{1}{u(u+1)}\, du + \int_0^1 f\left(\left\{ \frac{1}{u} \right\} \right) \left(1 - \frac{1}{u+1} \right) du.
\end{aligned}$$

Since $\{1/u\} = 1/u$ when $u > 1$, it remains to show

$$\int_0^1 f\left(\left\{ \frac{1}{u} \right\} \right) \frac{1}{u+1}\, du = \int_1^\infty f\left(\frac{1}{u} \right) \frac{1}{u(u+1)}\, du.$$

To prove this, we substitute $w = 1/u$ in the integral on the left side, and then, later, $u = 1/(w - j)$:

$$\begin{aligned}\int_0^1 f\left(\left\{\frac{1}{u}\right\}\right)\frac{1}{u+1}\,du &= \int_1^\infty f(\{w\})\frac{1}{w(w+1)}\,dw \\ &= \sum_{j=1}^\infty \int_j^{j+1} f(w-j)\frac{1}{w(w+1)}\,dv \\ &= \sum_{j=1}^\infty \int_1^\infty f\left(\frac{1}{u}\right)\frac{du}{(1+ju)(1+(j+1)u)} \\ &= \int_1^\infty f\left(\frac{1}{u}\right)\frac{1}{u}\sum_{j=1}^\infty\left(\frac{1}{1+ju} - \frac{1}{1+(j+1)u}\right)du \\ &= \int_1^\infty f\left(\frac{1}{u}\right)\frac{1}{u(u+1)}\,du.\end{aligned}$$

(**a**) By (**b**) and the asymptotic formula $H_n = \log n + \gamma + O(1/n)$ for the harmonic numbers H_n,

$$\begin{aligned}\int_0^1\int_0^1 \left\{\frac{x}{1-xy}\right\}dx\,dy = \int_0^1\left\{\frac{1}{x}\right\}dx &= \sum_{j=1}^\infty \int_{1/(j+1)}^{1/j}\left(\frac{1}{x} - j\right)dx \\ &= \lim_{n\to\infty}\sum_{j=1}^n\left(-\log j + \log(j+1) - \frac{1}{j+1}\right) \\ &= \lim_{n\to\infty}\left(\log(n+1) - (H_{n+1} - 1)\right) \\ &= \lim_{n\to\infty}\left(\log(n+1) - (\log(n+1) + \gamma - 1)\right) = 1 - \gamma.\end{aligned}$$

Editorial comment. The proposer and the GCHQ Problem Solving Group noted that when $k = 2$, the value of the integral in (**b**) is $\log(2\pi) - \gamma - 1$.

Also solved by K. F. Andersen (Canada), A. Berkane (Algeria), H. Chen, G. Fera, K. Gatesman, M. L. Glasser, J. A. Grzesik, O. Kouba (Syria), J. H. Lindsey II, Y. Mikayelyan (Armenia), T. Amdeberhan & V. H. Moll, P. Perfetti (Italy), N. C. Singer, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), L. Zhou, GCHQ Problem Solving Group (UK), and the proposer.

An Oscillating Binomial Sum

12032 [2018, 277]. *Proposed by David Galante (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas, Spain.* For a positive integer n, compute

$$\sum_{p=0}^n\sum_{k=p}^n(-1)^{k-p}\binom{k}{2p}\binom{n}{k}2^{n-k}.$$

Solution by Pierre Lalonde, Kingsey Falls, QC, Canada. The value is $2^{n/2}\cos(n\pi/4)$. Interchanging the order of summation converts the sum to

$$\sum_{k=0}^n(-1)^k\binom{n}{k}2^{n-k}\sum_{p=0}^k(-1)^p\binom{k}{2p}.$$

Since $(1 \pm i)^k = \sum_{p=0}^{k} (\pm 1)^p \binom{k}{p} i^p$, where $i = \sqrt{-1}$, cancellation in the binomial expansions yields

$$\frac{1}{2}\left((1+i)^k + (1-i)^k\right) = \sum_{p=0}^{k} i^{2p}\binom{k}{2p} = \sum_{p=0}^{k}(-1)^p\binom{k}{2p},$$

so the sum equals

$$\frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{n}{k}2^{n-k}\left((1+i)^k + (1-i)^k\right).$$

This sum contains the binomial expansions of $(2-(1+i))^n$ and $(2-(1-i))^n$, so the value is $\frac{1}{2}\left((1-i)^n + (1+i)^n\right)$. Finally, we compute

$$\begin{aligned}\frac{(1+i)^n + (1-i)^n}{2} &= \frac{\left(\sqrt{2}e^{i\pi/4}\right)^n + \left(\sqrt{2}e^{-i\pi/4}\right)^n}{2} \\ &= 2^{n/2}\,\frac{e^{n\pi i/4} + e^{-n\pi i/4}}{2} = 2^{n/2}\cos(n\pi/4).\end{aligned}$$

Also solved by U. Abel (Germany), T. Amdeberhan & V. H. Moll, K. F. Andersen (Canada), M. A. Carlton, R. Chapman (UK), P. P. Dálylay (Hungary), G. Fera (Italy), D. Fleischman, K. Gatesman, M. Jones, O. Kouba (Syria), K. T. L. Koo (China), O. P. Lossers (Netherlands), B. Lu, M. Omarjee (France), L. J. Peterson, R. Pratt, N. C. Singer, J. C. Smith, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), M. Wildon, L. Zhou, GCHQ Problem Solving Group (UK), and the proposers.

A Quadrilateral Inequality

12033 [2018, 277]. *Proposed by Dao Thanh Oai, Thai Binh, Vietnam, and Leonard Giugiuc, Drobeta Turnu Severin, Romania.* Let $ABCD$ be a convex quadrilateral with area S. Prove

$$AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2 \geq 8S + AB \cdot CD + BC \cdot AD - AC \cdot BD.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Ptolemy's inequality is $AB \cdot CD + BC \cdot AD \geq AC \cdot BD$. The AM-GM inequality then gives

$$2\,AC \cdot BD \leq 2(AB \cdot CD + BC \cdot AD) \leq AB^2 + CD^2 + BC^2 + AD^2 \tag{1}$$

and

$$2\,AC \cdot BD \leq AC^2 + BD^2. \tag{2}$$

Also,

$$0 \leq (AB - CD)^2 + (BC - AD)^2 + (AC - BD)^2. \tag{3}$$

Adding (1), (2), and (3) and dividing through by 2 yields

$$\begin{aligned}2\,AC \cdot BD \leq AB^2 + AC^2 + AD^2 + BC^2 &+ BD^2 + CD^2 \\ &- AB \cdot CD - BC \cdot AD - AC \cdot BD,\end{aligned}$$

which is equivalent to

$$\begin{aligned}AB \cdot CD + BC \cdot AD - AC \cdot BD &+ 4AC \cdot BD \\ &\leq AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2.\end{aligned} \tag{4}$$

The final step is to note that if θ is the angle between the diagonals AC and BD, then

$$S = \frac{1}{2}AC \cdot BD \cdot \sin\theta \le \frac{1}{2}AC \cdot BD. \tag{5}$$

The desired result follows from (4) and (5).

Equality holds when θ in (5) is a right angle and the right side of (3) is 0. These happen only when the quadrilateral is a square.

Editorial comment. Solvers Richard Stong and Li Zhou noted the stronger inequality

$$AB^2+AC^2+AD^2+BC^2+BD^2+CD^2 \ge 8S + 2\big(AB \cdot CD + BC \cdot AD - AC \cdot BD\big)$$

Also solved by E.. Bojaxhiu & E. Hysnelaj, P. P. Dályay (Hungary), D. Fleischman, K. Gatesman, H. Hyun (South Korea), K. T. L. Koo (China), V. Mikayelyan (Armenia), Davis Problem Solving Group, J. C. Smith, A. Stadler (Switzerland), R. Stong, B. Karaivanov (USA) & T. S. Vassilev (Canada), E. A. Weinstein, M. R. Yegan (Iran), L. Zhou, Davis Problem Solving Group, GCHQ Problem Solving Group (UK), and the proposer.

Multiples Without Large Digits

12034 [2018, 370]. *Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury J. Ionin, Central Michigan University, Mount Pleasant, MI.* Let N be any natural number that is not a multiple of 10. Prove that there is a multiple of N each of whose digits in base 10 is 1, 2, 3, 4, or 5.

Solution by Michael Reid, University of Central Florida, Orlando, FL. Let M be a natural number greater than 1, and let $L = M/q$, where q is the smallest prime divisor of M. As usual, let $[n] = \{1, \dots, n\}$. We prove the more general statement that every natural number N that is not a multiple of M has a multiple whose base M expansion has entries only in $[L]$. (In the given problem, $(M, q, L) = (10, 2, 5)$.)

Lemma 1. *If* $\gcd(N, M) = 1$*, then* N *divides* $\sum_{i=0}^{t} M^i$ *for some nonnegative* t.

Proof. With $a_s = \sum_{i=0}^{s} M^i$, by the pigeonhole principle some two numbers among $a_0, \dots, a_N$ are congruent modulo N. Since N divides their difference, which has the form $M^j a_t$, we see that N also divides a_t. ■

Lemma 2. *If* A *is a divisor of* M *such that* $\gcd(A, M/A) = 1$*, then the* A^k *numbers whose base-*M *expansions consist of* k *entries from* $[A]$ *are distinct modulo* A^k*. In particular, one of them is divisible by* A^k.

Proof. We use induction on k; the claim is trivial for $k = 1$. For $k \ge 1$, suppose that $\sum_{i=0}^{k} a_i M^i$ and $\sum_{i=0}^{k} b_i M^i$ are congruent modulo A^{k+1}. Since A divides M, the numbers $a_k M^k$ and $b_k M^k$ are divisible by A^k. Hence $\sum_{i=0}^{k-1} a^i M^i$ and $\sum_{i=0}^{k-1} b_i M_i$ are congruent modulo A^k. By the induction hypothesis, $a_i = b_i$ for $0 \le i \le k-1$. Subtracting the terms for $i < k$ from the assumed congruence leaves $a_k M^k \equiv b_k M^k \bmod A^{k+1}$. Thus A^{k+1} divides $(a_k - b_k)M^k$. Since A^k divides M^k, and M/A is relatively prime to A, we conclude that A divides $a_k - b_k$. Since $a_k, b_k \in [A]$, we have $a_k = b_k$. ■

Now let N be a positive integer not a multiple of M. For some prime p, the largest power p^b dividing N is less than the largest power p^c dividing M. Write N as $p^b RS$, where S is the largest divisor of N relatively prime to M. Thus every prime dividing R divides M, and $p \nmid R$.

Let $A = M/p^c$. Thus R divides some power of A, say A^k. Also A and M/A are relatively prime. By Lemma 2, R divides some number B whose base-M expansion consists of k entries from $[A]$.

Since S is relatively prime to M and thus also to M^k, Lemma 1 implies that S divides a number C of the form $\sum_{i=0}^{t}(M^k)^i$. Now BC is a multiple of RS, and the base-M expansion of BC consists of the expansion of B repeated $t+1$ times. Hence all the entries of this expansion lie in $[A]$. Finally, p^bBC is a multiple of p^bRS, which equals N. The entries in the base-M expansion of p^bBC are in $\{p^b, 2p^b, \ldots, Ap^b\}$, which is contained in $[L]$ since $Ap^b \leq M/p \leq M/q = L$.

Editorial comment. The restriction of entries to $[L]$ is in some sense sharp. If s is not a multiple of q, then sL is not divisible by M, and the units position of every multiple of sL is divisible by L and hence not in $[L-1]$.

On the other hand, when M is not squarefree, the set $[L]$ can be reduced to a proper subset. Suppose that M has prime factorization $\prod_{i=1}^{r} p_i^{e_i}$, and let $A_i = M/p_i^{e_i}$ for $i \in [r]$. The proof shows that every N not divisible by M has a multiple whose base-M expansion has all entries in the set $\bigcup_{i=1}^{r}\{p_i^{e_i-1}, 2p_i^{e_i-1}, \ldots, A_i p_i^{e_i-1}\}$, which is a proper subset of $[L]$ when M has a repeated prime factor.

For the original problem, several readers employed the Euler phi-function. In particular, when $\gcd(n, q-1) = 1$, the summed geometric series $\sum_{i=0}^{\phi(n)-1} q^i$ (a q-analogue of $\phi(n)$) is divisible by n, by Euler's theorem. For example, when $q = 10$ and $n = 77$, we have $\phi(77) = 60$, and hence 77 divides $\sum_{i=0}^{59} 10^i$.

Some substantial papers have been written about the digit distribution of multiples of integers. An example is Schmidt, W. M. (1983), The joint distribution of the digits of certain integer s-tuples, in Erdős, P., et al., eds., *Studies in Pure Mathematics: To the Memory of Paul Turán*, Basel: Birkhäuser, pp. 605–622.

Also solved by R. Chapman (UK), P. P. Dályay (Hungary), D. Fleischman, K. Gatesman, O. Geupel (Germany), E. J. Ionaşcu, D. Kim (South Korea), C. R. Pranesachar (India), A. Stadler (Switzerland), R. Stong, Y. Sun, R. Tauraso (Italy), M. Tetiva (Romania), GCHQ Problem Solving Group (UK), and the proposers.

Solving a Cubic to Minimize a Rational Expression

12035 [2018, 370]. *Proposed by Dinh Thi Nguyen, Tuy Hòa, Vietnam.* Find the minimum value of

$$(a^2+b^2+c^2)\left(\frac{1}{(3a-b)^2}+\frac{1}{(3b-c)^2}+\frac{1}{(3c-a)^2}\right)$$

as a, b, and c vary over all real numbers with $3a \neq b$, $3b \neq c$, and $3c \neq a$.

Solution by Li Zhou, Polk State College, Winter Haven, FL. Let $x = 3b - c$, $y = 3c - a$, and $z = 3a - b$. The hypothesis implies that x, y, and z are nonzero. The given expression becomes $F/52$ where

$$F = \left(4(x^2+y^2+z^2)+3(x+y+z)^2\right)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right).$$

To search for the minimum of F, it suffices to consider $x, y > 0$ and $z = -t < 0$. By the AM-GM inequality, $2(x^2+y^2) \geq (x+y)^2$ and

$$\frac{1}{x^2}+\frac{1}{y^2} \geq \frac{2}{xy} \geq \frac{8}{(x+y)^2},$$

with equality when $x = y$. Putting $x + y = s$, we then have

$$F \geq (2s^2+4t^2+3(s-t)^2)\left(\frac{8}{s^2}+\frac{1}{t^2}\right). \qquad (*)$$

Let r equal s/t, which is positive. The right side of (*) becomes

$$5r^2 - 6r + 47 - \frac{48}{r} + \frac{56}{r^2},$$

which we denote $f(r)$. Notice that $\lim_{r\to 0} f(r) = \lim_{r\to\infty} f(r) = \infty$ and

$$f'(r) = \frac{10r^4 - 6r^3 + 48r - 112}{r^3} = \frac{2(r+2)(5r^3 - 13r^2 + 26r - 28)}{r^3}.$$

According to the Cardano formula, the only positive zero ξ of $f'(r)$ is

$$\frac{13 + \sqrt[3]{4042 + 15\sqrt{120585}} + \sqrt[3]{4042 - 15\sqrt{120585}}}{15},$$

which is approximately 1.56431. Hence the required minimum value is $f(\xi)/52$, which is

$$\frac{2062 + \sqrt[3]{4420439038 + 12661425\sqrt{120585}} + \sqrt[3]{4420439038 - 12661425\sqrt{120585}}}{5460},$$

or approximately 0.8086454638.

Also solved by H. Chen, G. Fera, K. Gatesman, L. Giugiuc (Romania), O. Kouba (Syria), W.-K. Lai & J. Risher, K.-W. Lau (China), L. J. Peterson, M. Reid, J. C. Smith, A. Stadler (Switzerland), R. Stong, D. B. Tyler, and the proposer.

Metric Spaces with Few Isometry Types

12036 [2018, 370]. *Proposed by Greg Oman, University of Colorado, Colorado Springs, CO.* Two metric spaces (X, d) and (X', d') are said to be *isometric* if there is a bijection $\phi\colon X \to X'$ such that $d(a, b) = d'(\phi(a), \phi(b))$ for all $a, b \in X$. Let X be an infinite set. Find all metrics d on X such that (X, d) and (X', d') are isometric for every subset X' of X of the same cardinality as X. (Here, d' is the metric induced on X' by d.)

Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands. If $d(a, b)$ is independent of a and b when a and b differ, then (X, d) has the required property. We show that this is the only case. Suppose that at least two nonzero distances occur. Choose one of the distances, say δ, and define a coloring of the edges of the complete graph with vertex set X by letting xy be red if $d(x, y) = \delta$ and blue otherwise.

Given a point $p \in X$, let R be the set of neighbors of p via red edges, and let B be the set of neighbors of p via blue edges: $X = \{p\} \cup R \cup B$. Since X is infinite, R or B has the same cardinality as X; suppose it is B. Let $X' = X \setminus R = \{p\} \cup B$. Since X' has the same cardinality as X, by assumption the metric spaces (X, d) and (X', d') are isomorphic. Also the edge-colored complete graph on X and the induced one on X' are isomorphic. Since X' contains a vertex p incident only with blue edges, X also contains a vertex incident with only blue edges.

Let Y denote the subset of X consisting of all points incident with at least one red edge. The cardinality of Y must be smaller than the cardinality of X, because Y has no point incident only with blue edges. Finally, let $Y' = X \setminus Y$; the set Y' has the same cardinality as X. The graph induced by Y' has only blue edges, which implies that the original graph has only blue edges, contradicting our assumption.

The assumption that R has the same cardinality as X leads to a contradiction in the same way.

Editorial comment. Frederic Brulois and Gary Gruenhage provided a generalization: Let $\binom{X}{2}$ denote the family of 2-element subsets of X. Consider a function $f\colon \binom{X}{2} \to S$, where S is any set. If X is infinite and for any subset Y of X with the same cardinality as X there is a bijection $b\colon Y \to X$ such that $f(\{y_1, y_2\}) = f(\{b(y_1), b(y_2)\})$ for all $y_1, y_2 \in Y$, then f is a constant function.

Klaas Pieter Hart provided a different generalization: An infinite graph G that is isomorphic to all its induced subgraphs whose vertex sets have the same cardinality as G must be the complete graph or have no edge.

Also solved by F. Brulois, G. Gruenhage, J. W. Hagood, K. P. Hart (Netherlands), J. H. Lindsey II, A. Pathak, M. Reid, N. Sahoo, K. Schilling, R. Stong, and the proposer.

A Familiar Set Disguised

12037 [2018, 370]. *Proposed by José Manuel Rodríguez Caballero, Université du Québec, Montreal, QC, Canada.* For a positive integer n, let S_n be the set of pairs (a, k) of positive integers such that $\sum_{i=0}^{k-1}(a+i) = n$. Prove that the set

$$\left\{n : \sum_{(a,k)\in S_n} (-1)^{a-k} \neq 0\right\}$$

is closed under multiplication.

Solution by GCHQ Problem Solving Group, Cheltenham, UK. Let A be the set defined in the problem statement. Each $(a, k) \in S_n$ satisfies

$$n = ka + \sum_{i=0}^{k-1} i = ka + \frac{k(k-1)}{2},$$

and thus

$$2n = k(k + 2a - 1).$$

The factors k and $k + 2a - 1$ have opposite parity, and also $k + 2a - 1 > k$. Given n, we can generate a pair $(a, k) \in S_n$ by writing $2n = E \times O$, where E is even and O is odd, and setting $k = \min(E, O)$ and $a = (|O - E| + 1)/2$. The process is reversible, so we have a bijection from S_n to the set of even/odd factorizations $2n = E \times O$. We write these as $2n = (2^T u) \times v$, where u and v are both odd.

Note also that $a + k = (E + O + 1)/2$. If $(2^T u) + v \equiv 1 \pmod 4$, then $a + k$ is odd, while if $(2^T u) + v \equiv 3 \pmod 4$, then $a + k$ is even. Because $(-1)^{a-k} = (-1)^{a+k}$, we have $n \in A$ if and only if the number of even/odd factorizations resulting in $a + k$ even is different from the number resulting in $a + k$ odd.

Let p be a prime factor of n. Switching p from u to v or vice versa does not change the congruence class of $2^T u$ or v modulo 4 if $p \equiv 1 \pmod 4$. However, if $p \equiv 3 \pmod 4$, then the switch changes the sign of v and leaves the congruence class of $2^T u$ unchanged, so it changes the class of $(2^T u) + v$.

If some prime factor p congruent to 3 modulo 4 occurs in $2n$ with odd power, then for any fixed distribution of the other factors, there are the same number of factorizations in which p contributes an even number or an odd number of factors to v. Hence there are the same number of factorizations with $a + k$ even or odd, and $n \notin A$.

Conversely, suppose that all such prime factors occur with even power. When all the odd prime factors are in v, and $u = 1$, we have $(2^T u) + v \equiv 2^T + 1 \pmod 4$, and the class depends on whether $T > 1$. The class remains the same for any distribution of the prime factors congruent to 1 modulo 4. Thus we need only consider multisets of the prime factors congruent to 3 modulo 4, where the bound on the multiplicity of each is even. With an even bound, the number of choices for the multiplicity of each such factor is odd. Hence there are an odd number of multisets of the prime factors congruent to 3 modulo 4. With an odd number of choices, there cannot be the same number with even size as odd size. Hence there will not be the same number of factorizations with $a + k$ even and odd, and so $n \in A$.

Since the product of two numbers whose prime factorizations have each prime factor congruent to 3 modulo 4 occurring with even power also has the same property, A is closed under multiplication.

Editorial comment. Several solvers noted that A is the set of all positive integers that can be expressed as a sum of two squares.

Also solved by R. Chapman (UK), K. Gatesman, E. J. Ionaşcu, P. Lalonde (Canada), O. P. Lossers (Netherlands), J. C. Smith, and the proposer.

An Inequality with Medians

12038 [2018, 370]. *Proposed by George Apostolopoulos, Messolonghi, Greece.* Let ABC be an acute triangle with sides of length a, b, and c opposite angles A, B, and C, respectively, and with medians of length m_a, m_b, and m_c emanating from A, B, and C, respectively. Prove

$$\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} \geq 9 \cos A \, \cos B \, \cos C.$$

Solution by Subhankar Gayen, Vivekananda Mission Mahavidyalaya, India. Let M be the midpoint of BC. Suppose that AM intersects the circumcircle of $\triangle ABC$ at D. By the power-of-the-point theorem, $m_a \cdot MD = a^2/4$, and two applications of the law of cosines yields $a^2/4 = (b^2+c^2)/2 - m_a^2$. Hence $b^2 + c^2 = 2m_a\,(m_a + MD)$. Since AD is a chord of the circumcircle, $m_a + MD \leq 2R$, where R is the circumradius of $\triangle ABC$. Hence $4Rm_a \geq b^2 + c^2$. Using this and the two other analogous inequalities yields

$$\begin{aligned}\frac{m_a^2}{b^2+c^2} + \frac{m_b^2}{c^2+a^2} + \frac{m_c^2}{a^2+b^2} &\geq \frac{b^2+c^2}{16R^2} + \frac{c^2+a^2}{16R^2} + \frac{a^2+b^2}{16R^2} \\ &= \frac{a^2+b^2+c^2}{8R^2} \\ &= \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2} \\ &= 1 + \cos A \cos B \cos C,\end{aligned}$$

where we have used the generalized law of sines in the second-to-last step and $A + B + C = \pi$ to obtain the last equality.

We complete the proof by showing that $1 \geq 8 \cos A \cos B \cos C$. This follows from $\cos(x)\cos(y) < \cos^2((x+y)/2)$ when $x \neq y$, because this last inequality shows that $\cos A \cos B \cos C$ cannot take its maximum value on a triangle ABC unless $A = B = C = \pi/3$.

Note that the assumption that $\triangle ABC$ is acute is unnecessary and also that equality holds only when $\triangle ABC$ is equilateral.

Also solved by H. Bailey, M. Bataille (France), H. Chen, G. Fera, L. Giugiuc (Romania), W. Janous (Austria), O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, J. F. Loverde, M. Lukarevski (Macedonia), P. Nüesch (Switzerland), P. Perfetti (Italy), C. R. Pranesachar (India), V. Schindler (Germany), D. Smith (Canada), J. C. Smith, A. Stadler (Switzerland), R. Stong, M. Vowe (Switzerland), T. Wiandt, M. R. Yegan (Iran), L. Zhou, T. Zvonaru (Romania), GCHQ Problem Solving Group (UK), and the proposer.